

Abelian Decomposition of General Relativity

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Based on the view that Einstein's theory can be interpreted as a gauge theory of Lorentz group, we decompose the gravitational connection (the gauge potential of Lorentz group) Γ_μ into the restricted connection made of the potential of the maximal Abelian subgroup H of Lorentz group G and the valence connection made of G/H part of the potential which transforms covariantly under Lorentz gauge transformation. With this decomposition we show that the Einstein's theory can be decomposed into the restricted part made of the restricted connection which has the full Lorentz gauge invariance and the valence part made of the valence connection which plays the role of gravitational source of the restricted gravity. We show that there are two different Abelian decomposition of Einstein's theory, the light-like (or null) decomposition and the non light-like (or non-null) decomposition, because Lorentz group has two maximal Abelian subgroups. In this decomposition the role of the metric $g_{\mu\nu}$ is replaced by a four-index metric tensor $\mathbf{g}_{\mu\nu}$ which transforms covariantly under the Lorentz group, and the metric-compatibility condition $\nabla_\alpha g_{\mu\nu} = 0$ of the connection is replaced by the gauge and generally covariant condition $\mathcal{D}_\mu \mathbf{g}^{\mu\nu} = 0$. The decomposition shows the existence of a restricted theory of gravitation which has the full general invariance but is much simpler and has less physical degrees of freedom than Einstein's theory. Moreover, it tells that the restricted gravity can be written as an Abelian gauge theory, which implies that the graviton can be described by a massless spin-one field.

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I. INTRODUCTION

Einstein's theory of gravitation and the gauge theory of electroweak and strong interactions are two fundamental ingredients of theoretical physics which describe all known interactions of nature. But they are closely related to each other. The gauge theory can be viewed as a part of Einstein's theory originating from the extrinsic curvature of higher-dimensional unified space [1, 2]. It is well-known that the $(4+n)$ -dimensional unified space made of the 4-dimensional space-time and an n -dimensional internal space, the $(4+n)$ -dimensional Einstein's theory reproduces the gauge theory when the internal space has an n -dimensional isometry G . In fact the $(4+n)$ -dimensional Einstein's theory provides a natural unification of gauge theory with gravitation, which is known as the Kaluza-Klein miracle [2, 3].

Conversely Einstein's theory itself can be understood as a gauge theory, because the general invariance of Ein-

stein's theory can be viewed as a gauge invariance [4, 5]. One can view it as a gauge theory of 4-dimensional translation group, because the local 4-dimensional translation can be identified as the general coordinate transformation. In this case one can identify the gauge potential of the translation group as the (non-trivial part of the) tetrad [5, 6]. Or, one can view it as a gauge theory of Lorentz group (or Poincare group in general), because the Lorentz gauge transformation can also be interpreted as the general coordinate transformation. In this case one can identify the gauge potential of Lorentz group as the spin connection [7, 8]. This confirms that the two theories are closely related.

During the last few decades our understanding of non-Abelian gauge theory has been extended very much. By now it has been well known that the non-Abelian gauge theory allows the Abelian decomposition [9, 10]. The non-Abelian gauge potential can be decomposed into the restricted potential of the maximal Abelian subgroup H of the gauge group G which has an electric-magnetic duality and the valence potential of G/H which transforms covariantly under G . A remarkable feature of this decomposition is that it is gauge independent. As importantly, the restricted potential has the full non-Abelian gauge

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degrees of freedom, in particular the topological degrees of the gauge group G , in spite of the fact that it consists of only the Abelian degrees of the maximal Abelian subgroup H . This means that we can construct a restricted gauge theory, a non-Abelian gauge theory made of only the restricted potential which has much less physical degrees of freedom, which nevertheless has the full gauge invariance. Moreover, we can recover the full non-Abelian gauge theory simply by adding the valence part. This tells that the non-Abelian gauge theory can be interpreted as a restricted gauge theory which has the valence potential as the gauge covariant source [9, 10]. The importance of this decomposition is that the restricted part plays a crucial role in non-Abelian dynamics, in particular in the confinement mechanism in QCD [11–14].

The main purpose of this paper is to discuss a similar Abelian decomposition of Einstein's theory. Regarding the theory as a gauge theory of Lorentz group and applying the Abelian decomposition to the gauge potential of Lorentz group, we first show that we can decompose the gravitational connection to the restricted connection and the valence connection. With this we decompose the Einstein's theory into the restricted part made of the restricted connection and the valence part made of the gauge covariant valence connection. We show that Einstein's theory allows two different Abelian decompositions, light-like decomposition and non light-like decomposition. This is because the Lorentz group has two maximal Abelian subgroups. With the Abelian decomposition we finally show that the restricted gravity can be interpreted as an Abelian gauge theory. *Our analysis tells that the Einstein's theory can be viewed as a restricted theory of gravitation which has the gauge covariant valence connection as the gravitational source. More importantly our analysis implies that the graviton can be described by a massless spin-one gauge potential.*

To decompose the Einstein's theory we introduce the gauge covariant metric $\mathbf{g}_{\mu\nu}$, an antisymmetric $(0, 2)$ -tensor in space-time $g_{\mu\nu}^{ab}$ which forms an adjoint representation of Lorentz group, and show that the metric-compatibility condition of the gravitational connection $\nabla_\alpha g_{\mu\nu} = 0$ is transformed to the gauge covariant (Lorentz covariant) and generally covariant condition $\mathcal{D}_\mu \mathbf{g}^{\mu\nu} = 0$ which assures the invariance of $\mathbf{g}_{\mu\nu}$ under the parallel transport along the ∂_μ -direction.

Of course, Einstein's theory as a gauge theory of Lorentz group is different from the ordinary non-Abelian gauge theory. In gauge theory the fundamental field is the gauge potential, but in Einstein's theory the fundamental field is the metric. And in the gauge formulation the gauge potential of Lorentz group corresponds to the gravitational connection, not the metric. Also, in gauge theory the Yang-Mills Lagrangian is quadratic in field strength. But in gravitation the Einstein-Hilbert Lagrangian is made of the scalar curvature, which is linear in field strength [7]. Nevertheless we can still make the Abelian decomposition of the gravitational connection,

and express the Einstein-Hilbert Lagrangian in terms of the restricted connection and the valence connection. With this we can separate the restricted part of gravitation from the Einstein's theory, and show that the theory can be interpreted as a restricted theory of gravity which has the valence connection as the gravitational source.

The paper is organized as follows. In Section II we review the prototype Abelian decomposition, the $U(1)$ decomposition of $SU(2)$ gauge theory, as an example to help us to understand the Abelian decomposition of Einstein's theory. In Section III we show how to decompose the gravitational connection to the Abelian part and the valence part. We show that there are two different ways of Abelian decomposition, because the Lorentz group has two maximal Abelian subgroups. In Section IV introduce the concept of the Lorentz covariant metric tensor, and show how to decompose the Einstein's theory to the restricted part and the valence part. We discuss two different Abelian decompositions of Einstein's theory separately. In section V we introduce two restricted gravities based on two Abelian decompositions, and show that they can be described by an Abelian gauge theory. In particular we argue that the graviton can be described by a massless spin-one gauge field. Finally in Section VI we discuss the physical implications of our results.

II. ABELIAN DECOMPOSITION OF $SU(2)$: A REVIEW

To understand how the Abelian decomposition works in Einstein's theory, it is important to understand the Abelian decomposition $SU(2)$ gauge theory for two reasons. First, it is the simplest non-Abelian gauge theory in which we can demonstrate the Abelian decomposition. But more importantly, it is the rotation subgroup of Lorentz group, so that the Abelian decomposition of $SU(2)$ directly applies to the Abelian decomposition of Einstein's theory. For these reasons we review the Abelian decomposition $SU(2)$ gauge theory first [9, 10].

Let \hat{n} be an arbitrary isotriplet unit vector field of $SU(2)$, and identify the maximal Abelian subgroup to be the $U(1)$ subgroup which leaves \hat{n} invariant. Clearly \hat{n} selects the “Abelian” direction (i.e., the color charge direction) at each space-time point, and the the Abelian magnetic isometry can be described by the following constraint equation

$$D_\mu \hat{n} = \partial_\mu \hat{n} + g \vec{A}_\mu \times \hat{n} = 0. \quad (\hat{n}^2 = 1) \quad (1)$$

This has the unique solution for \vec{A}_μ which defines the restricted potential \hat{A}_μ which leaves \hat{n} invariant under the parallel transport,

$$\hat{A}_\mu = A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \quad (2)$$

where $A_\mu = \hat{n} \cdot \vec{A}_\mu$ is the “electric” potential. This process of selecting the restricted potential is called the Abelian projection [9, 10].

With the Abelian projection we can retrieve the full gauge potential by adding the gauge covariant valence potential \vec{X}_μ to the restricted potential,

$$\begin{aligned} \vec{A}_\mu &= A_\mu \hat{n} - \frac{1}{g} \hat{n} \times \partial_\mu \hat{n} + \vec{X}_\mu = \hat{A}_\mu + \vec{X}_\mu, \\ (\hat{n}^2 &= 1, \quad \hat{n} \cdot \vec{X}_\mu = 0). \end{aligned} \quad (3)$$

This is the Abelian decomposition which decomposes the gauge potential into the restricted potential \hat{A}_μ and the valence potential \vec{X}_μ [9, 10].

Let $\vec{\alpha}$ is an infinitesimal gauge parameter. Under the infinitesimal gauge transformation

$$\delta \hat{n} = -\vec{\alpha} \times \hat{n}, \quad \delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha}, \quad (4)$$

one has

$$\begin{aligned} \delta A_\mu &= \frac{1}{g} \hat{n} \cdot \partial_\mu \vec{\alpha}, \quad \delta \hat{A}_\mu = \frac{1}{g} \hat{D}_\mu \vec{\alpha}, \\ \delta \vec{X}_\mu &= -\vec{\alpha} \times \vec{X}_\mu. \end{aligned} \quad (5)$$

This shows that \hat{A}_μ by itself describes an $SU(2)$ connection which enjoys the full $SU(2)$ gauge degrees of freedom. Furthermore \vec{X}_μ transforms covariantly under the gauge transformation. Most importantly, the decomposition is gauge-independent. Once the color direction \hat{n} is selected, the decomposition follows independent of the choice of a gauge. This decomposition was first introduced long time ago in an attempt to demonstrate the monopole condensation in QCD [9, 10]. But recently the importance of the decomposition in the non-Abelian dynamics has been emphasized by many authors [11, 14].

In particular, recently the Abelian decomposition has been successfully used in the lattice calculation of QCD to demonstrate the monopole condensation and color confinement in a gauge independent way [15]. A critical defect of the conventional lattice calculations is that the calculation is gauge dependent, because one has to choose a gauge (so-called the maximally Abelian gauge) to perform the calculation. With the Abelian decomposition, however, one does not have to choose a gauge to perform the calculation. So the recent calculation was able to demonstrate that the monopole condensation in QCD is a gauge independent phenomenon.

To understand the physical meaning of our decomposition notice that the restricted potential \hat{A}_μ actually has a dual structure. Indeed the field strength made of the restricted potential is decomposed as

$$\begin{aligned} \hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + g \hat{A}_\mu \times \hat{A}_\nu = (F_{\mu\nu} + H_{\mu\nu}) \hat{n}, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ H_{\mu\nu} &= -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu, \end{aligned} \quad (6)$$

where \tilde{C}_μ is the “magnetic” potential [9, 10]. Notice that we can always introduce the magnetic potential (at least locally section-wise), because $H_{\mu\nu}$ forms a closed two-form

$$\partial_\mu H_{\mu\nu}^d = 0 \quad (H_{\mu\nu}^d = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} H_{\rho\sigma}). \quad (7)$$

This allows us to identify the non-Abelian magnetic potential by

$$\vec{C}_\mu = -\frac{1}{g} \hat{n} \times \partial_\mu \hat{n}, \quad (8)$$

in terms of which the magnetic field strength is expressed as

$$\begin{aligned} \vec{H}_{\mu\nu} &= \partial_\mu \vec{C}_\nu - \partial_\nu \vec{C}_\mu + g \vec{C}_\mu \times \vec{C}_\nu \\ &= -g \vec{C}_\mu \times \vec{C}_\nu = -\frac{1}{g} \partial_\mu \hat{n} \times \partial_\nu \hat{n} = H_{\mu\nu} \hat{n}. \end{aligned} \quad (9)$$

As importantly \hat{A}_μ , as an $SU(2)$ potential, retains all the essential topological characteristics of the original non-Abelian potential. This is because the topological field \hat{n} naturally represents the non-Abelian topology $\pi_2(S^2)$ which describes the mapping from an S^2 in 3-dimensional space R^3 to the coset space $SU(2)/U(1)$, and $\pi_3(S^3) \simeq \pi_3(S^2)$ which describes the mapping from the compactified 3-dimensional space S^3 to the group space S^3 . Clearly the isolated singularities of \hat{n} defines $\pi_2(S^2)$ which describes the non-Abelian monopoles. Indeed \vec{C}_μ with $\hat{n} = \hat{r}$ describes precisely the Wu-Yang monopole [9, 16]. This is why we call \vec{C}_μ the magnetic potential. Besides, with the S^3 compactification of R^3 , \hat{n} characterizes the Hopf invariant $\pi_3(S^2) \simeq \pi_3(S^3)$ which describes the topologically distinct vacua [17–19].

With (3) we have

$$\vec{F}_{\mu\nu} = \hat{F}_{\mu\nu} + \hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu + g \vec{X}_\mu \times \vec{X}_\nu, \quad (10)$$

so that the Yang-Mills Lagrangian is expressed as

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} \vec{F}_{\mu\nu}^2 = -\frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{4} (\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu)^2 \\ &\quad - \frac{g}{2} \hat{F}_{\mu\nu} \cdot (\vec{X}_\mu \times \vec{X}_\nu) - \frac{g^2}{4} (\vec{X}_\mu \times \vec{X}_\nu)^2 \\ &\quad + \lambda (\hat{n}^2 - 1) + \lambda_\mu \hat{n} \cdot \vec{X}_\mu, \end{aligned} \quad (11)$$

where λ and λ_μ are the Lagrangian multipliers. From the Lagrangian we have

$$\begin{aligned} \delta A_\nu &: \partial_\mu (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) \\ &= -g \hat{n} \cdot \{ \vec{X}_\mu \times (\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu) \}, \\ \delta \vec{X}_\nu &: \hat{D}_\mu (\hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu) \\ &= g (F_{\mu\nu} + H_{\mu\nu} + X_{\mu\nu}) \hat{n} \times \vec{X}_\mu, \\ X_{\mu\nu} &= g \hat{n} \cdot (\vec{X}_\mu \times \vec{X}_\nu). \end{aligned} \quad (12)$$

Notice that here \hat{n} has no equation of motion even though the Lagrangian contains it explicitly. This is because it

represents a topological degrees of freedom, not a local degrees of freedom [9, 10]. From this we conclude that the non-Abelian gauge theory can be viewed as a restricted gauge theory made of the restricted potential, which has an additional colored source made of the valence gluon.

Obviously the Lagrangian (11) is invariant under the active gauge transformation (4). But notice that the decomposition introduces another gauge symmetry that we call the passive gauge transformation [12, 13],

$$\delta \hat{n} = 0, \quad \delta \vec{A}_\mu = \frac{1}{g} D_\mu \vec{\alpha}, \quad (13)$$

under which we have

$$\begin{aligned} \delta A_\mu &= \frac{1}{g} \hat{n} \cdot D_\mu \vec{\alpha}, & \delta \hat{A}_\mu &= \frac{1}{g} (\hat{n} \cdot D_\mu \vec{\alpha}) \hat{n}, \\ \delta \vec{X}_\mu &= \frac{1}{g} \{D_\mu \vec{\alpha} - (\hat{n} \cdot D_\mu \vec{\alpha}) \hat{n}\}. \end{aligned} \quad (14)$$

This is because, for a given \vec{A}_μ , one can have infinitely many different decomposition of (3), with different \hat{A}_μ and \vec{X}_μ choosing different \hat{n} . Equivalently, for a fixed \hat{n} , one can have infinitely many different \vec{A}_μ which are gauge-equivalent to each other. So our decomposition automatically induce another type of gauge invariance which comes from different choices of decomposition. This extra gauge invariance plays a crucial role in quantizing the theory [10].

An important advantage of the decomposition (3) is that it can actually “Abelianize” (or more precisely “dualize”) the non-Abelian dynamics, without any gauge fixing [9, 12]. To see this let $(\hat{n}_1, \hat{n}_2, \hat{n}_3 = \hat{n})$ be a right-handed orthonormal basis and let

$$\begin{aligned} \vec{X}_\mu &= X_\mu^1 \hat{n}_1 + X_\mu^2 \hat{n}_2, \\ (X_\mu^1 &= \hat{n}_1 \cdot \vec{X}_\mu, \quad X_\mu^2 = \hat{n}_2 \cdot \vec{X}_\mu) \end{aligned}$$

and find

$$\begin{aligned} \hat{D}_\mu \vec{X}_\nu &= \{\partial_\mu X_\nu^1 - g(A_\mu + \tilde{C}_\mu) X_\nu^2\} \hat{n}_1 \\ &+ \{\partial_\mu X_\nu^2 + g(A_\mu + \tilde{C}_\mu) X_\nu^1\} \hat{n}_2, \end{aligned} \quad (15)$$

where now the magnetic potential \tilde{C}_μ can be written explicitly as

$$\tilde{C}_\mu = -\frac{1}{g} \vec{n}_1 \cdot \partial_\mu \vec{n}_2, \quad (16)$$

up to the $U(1)$ gauge transformation which leaves \hat{n} invariant. So with

$$\begin{aligned} \bar{A}_\mu &= A_\mu + \tilde{C}_\mu, \quad \bar{F}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu, \\ X_\mu &= \frac{1}{\sqrt{2}} (X_\mu^1 + i X_\mu^2), \end{aligned} \quad (17)$$

one could express the Lagrangian explicitly in terms of the dual potential B_μ and the complex vector field X_μ ,

$$\mathcal{L} = -\frac{1}{4} \bar{F}_{\mu\nu}^2 - \frac{1}{2} |\bar{D}_\mu X_\nu - \bar{D}_\nu X_\mu|^2 + ig \bar{F}_{\mu\nu} X_\mu^* X_\nu$$

$$\begin{aligned} &-\frac{1}{2} g^2 \{(X_\mu^* X_\mu)^2 - (X_\mu^*)^2 (X_\nu)^2\}, \\ &\bar{D}_\mu = \partial_\mu + ig \bar{A}_\mu. \end{aligned} \quad (18)$$

Clearly this describes an Abelian gauge theory coupled to the charged vector field X_μ . But the important point here is that the Abelian potential \bar{A}_μ is given by the sum of the electric and magnetic potentials $A_\mu + \tilde{C}_\mu$. In this form the equations of motion (12) is re-expressed as

$$\begin{aligned} \partial_\mu (\bar{F}_{\mu\nu} + X_{\mu\nu}) &= ig X_\mu^* (\bar{D}_\mu X_\nu - \bar{D}_\nu X_\mu) \\ &- ig X_\mu (\bar{D}_\mu X_\nu - \bar{D}_\nu X_\mu)^*, \\ \bar{D}_\mu (\bar{D}_\mu X_\nu - \bar{D}_\nu X_\mu) &= ig X_\mu (\bar{F}_{\mu\nu} + X_{\mu\nu}), \\ X_{\mu\nu} &= -ig (X_\mu^* X_\nu - X_\nu^* X_\mu). \end{aligned} \quad (19)$$

This shows that one can indeed Abelianize the non-Abelian theory with our decomposition. The remarkable change in this “Abelian” formulation is that here the topological field \hat{n} is replaced by the magnetic potential \tilde{C}_μ .

III. ABELIAN DECOMPOSITION OF GRAVITATIONAL CONNECTION

We can apply the above Abelian decomposition to Einstein’s theory, regarding Einstein’s theory as a gauge theory of Lorentz group. To do this we introduce a coordinate basis

$$[\partial_\mu, \partial_\nu] = 0, \quad (\mu, \nu = t, x, y, z)$$

and an orthonormal basis

$$\begin{aligned} [\xi_a, \xi_b] &= f_{ab}^c \xi_c, \quad (a, b = 0, 1, 2, 3) \\ \xi_a &= e_a^\mu \partial_\mu, \quad \partial_\mu = e_\mu^a \xi_a, \end{aligned} \quad (20)$$

where e_μ^a and e_a^μ are the tetrad and inverse tetrad. Let $J_{ab} = -J_{ba}$ be the generators of Lorentz group,

$$\begin{aligned} [J_{ab}, J_{cd}] &= \eta_{ac} J_{bd} - \eta_{bc} J_{ad} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} \\ &= f_{ab,cd}^{mn} J_{mn}, \\ f_{ab,cd}^{mn} &= \eta_{ac} \delta_b^{[m} \delta_d^{n]} - \eta_{bc} \delta_a^{[m} \delta_d^{n]} \\ &+ \eta_{bd} \delta_a^{[m} \delta_c^{n]} - \eta_{ad} \delta_b^{[m} \delta_c^{n]}, \end{aligned} \quad (21)$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Clearly J_{ab} has the following 4-dimensional matrix representation

$$(J_{ab})_c^d = -\eta_{ac} \delta_b^d + \eta_{bc} \delta_a^d, \quad (22)$$

so that under the infinitesimal gauge transformation we have

$$\delta e_\mu^c = (\eta_{ad} \delta_b^c - \eta_{bd} \delta_a^c) \alpha^{ab} e_\mu^d, \quad (23)$$

where $\alpha^{ab} (= -\alpha^{ba})$ is an infinitesimal gauge parameter of the Lorentz group. Instead of (ab, cd, \dots) we can use the

index $(A, B, \dots) = (1, 2, 3, 4, 5, 6) = (23, 31, 12, 01, 02, 03)$, and write

$$[J_A, J_B] = f_{AB}^C J_C.$$

Moreover, with

$$\begin{aligned} L_{1,2,3} &= J_{1,2,3} = J_{23,31,12} \\ K_{1,2,3} &= J_{4,5,6} = J_{01,02,03} \end{aligned}$$

the Lorentz algebra is written as

$$\begin{aligned} [L_i, L_j] &= \epsilon_{ijk} L_k, & [L_i, K_j] &= \epsilon_{ijk} K_k, \\ [K_i, K_j] &= -\epsilon_{ijk} L_k, & (i, j, k &= 1, 2, 3) \end{aligned} \quad (24)$$

where L_i and K_i are the 3-dimensional rotation and boost generators. Notice that the generators can be viewed as the left-invariant basis vector fields on the Lorentz group manifold which satisfy the commutation relation.

As we have pointed out, we can regard Einstein's theory as a gauge theory of Lorentz group. In this view the gravitational connection $\Gamma_{\mu\nu}^\rho$ (or more precisely the spin connection ω_μ^{ab}) corresponds to the gauge potential Γ_μ^{ab} , and the curvature tensor $R_{\mu\nu}^{ab}$ corresponds to the gauge field strength $F_{\mu\nu}^{ab}$ of Lorentz group. And to obtain the desired decomposition we have to decompose the gauge potential Γ_μ^{ab} first. Now, to apply the above $SU(2)$ decomposition to Lorentz group, we have to keep in mind that there are notable differences between $SU(2)$ and Lorentz group. First, the Lorentz group is non-compact, so that the invariant metric is indefinite. Secondly, the Lorentz group has the well-known invariant tensor ϵ_{abcd} which allows the dual transformation. Thirdly, the Lorentz group has rank two, so that it has two commuting Abelian subgroups and two Casimir invariants. Finally, the Lorentz group has two different maximal Abelian subgroups A_2 and B_2 [20]. These differences make the decomposition more complicated.

The invariant metric δ_{AB} of Lorentz group is given by

$$\begin{aligned} \delta_{AB} &= -\frac{1}{4} f_{AC}^D f_{BD}^C \\ &= \text{diag} (+1, +1, +1, -1, -1, -1). \end{aligned} \quad (25)$$

Let p^{ab} ($p^{ab} = -p^{ba}$) (or p^A) be a gauge covariant sextet vector which forms an adjoint representation of Lorentz group,

$$\delta p^{cd} = -\frac{1}{2} f_{ab,mn}^{cd} \alpha^{ab} p^{mn}. \quad (26)$$

Clearly p^{ab} can be understood as an anti-symmetric tensor in 4-dimensional Minkowski space which can be expressed by two 3-dimensional vectors \vec{m} and \vec{e} , which transform exactly like the magnetic and electric components of an electromagnetic tensor under the 4-dimensional Lorentz transformation. And we denote p^{ab}

by \mathbf{p} ,

$$\begin{aligned} \mathbf{p} &= \frac{1}{2} p_{ab} \mathbf{I}^{ab} = \begin{pmatrix} \vec{m} \\ \vec{e} \end{pmatrix}, & p^{ab} &= \mathbf{p} \cdot \mathbf{I}^{ab} = \frac{1}{2} p^{mn} I_{mn}^{ab}, \\ \mathbf{I}^{ab} &= \begin{pmatrix} \hat{m}^{ab} \\ \hat{e}^{ab} \end{pmatrix}, \\ \hat{m}_i^{ab} &= \epsilon_{0i}^{ab}, & \hat{e}_i^{ab} &= (\delta_0^a \delta_i^b - \delta_0^b \delta_i^a), \\ I_{mn}^{ab} &= (\delta_m^a \delta_n^b - \delta_m^b \delta_n^a) = -(J_{mn})^{ab}. \end{aligned} \quad (27)$$

where $m_i = \epsilon_{ijk} p^{jk}/2$ ($i, j, k = 1, 2, 3$) is the magnetic (or rotation) part and $e_i = p^{0i}$ is the electric (or boost) part of \mathbf{p} . From the invariant metric (25) we have

$$\mathbf{p}^2 = \frac{1}{2} p_{ab} p^{ab} = \vec{m}^2 - \vec{e}^2, \quad (28)$$

so that the invariant length can be positive, zero, or negative. This, of course, is due to the fact that the invariant metric (25) is indefinite.

The Lorentz group has another important invariant tensor ϵ_{AB} which comes from the totally anti-symmetric invariant tensor ϵ_{abcd} ,

$$\epsilon_{AB} = \epsilon_{ab,cd} = \epsilon_{abcd}. \quad (29)$$

This tells that any adjoint representation of Lorentz group has its dual partner. In particular, \mathbf{p} has the dual vector $\tilde{\mathbf{p}}$ defined by $\tilde{p}^{ab} = \epsilon^{abcd} p_{cd}/2$. With (27) we have (with $\epsilon_{0123} = +1$)

$$\begin{aligned} \tilde{\mathbf{p}} &= \begin{pmatrix} \vec{e} \\ -\vec{m} \end{pmatrix}, & \tilde{\tilde{\mathbf{p}}} &= -\mathbf{p}, \\ \tilde{\mathbf{p}}^2 &= \vec{e}^2 - \vec{m}^2 = -\mathbf{p}^2, \\ \mathbf{p} \cdot \tilde{\mathbf{p}} &= \frac{1}{4} \epsilon_{abcd} p^{ab} p^{cd} = 2\vec{m} \cdot \vec{e}. \end{aligned} \quad (30)$$

Moreover, we have

$$[p, \tilde{p}] = 0, \quad \mathbf{p} \times \tilde{\mathbf{p}} = 0. \quad (31)$$

This tells that any two vectors which are dual to each other are always commuting. Finally we have the following vector operations,

$$\begin{aligned} \mathbf{p} \cdot \mathbf{p}' &= \vec{m} \cdot \vec{m}' - \vec{e} \cdot \vec{e}', \\ \mathbf{p} \cdot \tilde{\mathbf{p}}' &= \vec{m} \cdot \vec{e}' + \vec{e} \cdot \vec{m}' = \tilde{\mathbf{p}} \cdot \mathbf{p}', \\ \mathbf{p} \times \mathbf{p}' &= \begin{pmatrix} \vec{m} \times \vec{m}' - \vec{e} \times \vec{e}' \\ \vec{m} \times \vec{e}' + \vec{e} \times \vec{m}' \end{pmatrix} = -\tilde{\mathbf{p}} \times \tilde{\mathbf{p}}', \\ \mathbf{p} \times \tilde{\mathbf{p}}' &= \begin{pmatrix} \vec{m} \times \vec{e}' + \vec{e} \times \vec{m}' \\ -\vec{m} \times \vec{m}' + \vec{e} \times \vec{e}' \end{pmatrix} = \tilde{\mathbf{p}} \times \mathbf{p}', \\ \widetilde{\mathbf{p} \times \mathbf{p}'} &= \mathbf{p} \times \tilde{\mathbf{p}}' = \tilde{\mathbf{p}} \times \mathbf{p}', \\ \mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3) &= \mathbf{p}_2 \cdot (\mathbf{p}_3 \times \mathbf{p}_1) = \mathbf{p}_3 \cdot (\mathbf{p}_1 \times \mathbf{p}_2), \\ \mathbf{p}_1 \times (\mathbf{p}_2 \times \mathbf{p}_3) &= [\mathbf{p}_2 (\mathbf{p}_1 \cdot \mathbf{p}_3) - \mathbf{p}_3 (\mathbf{p}_1 \cdot \mathbf{p}_2)] \\ &\quad - [\tilde{\mathbf{p}}_2 (\mathbf{p}_1 \cdot \tilde{\mathbf{p}}_3) - \tilde{\mathbf{p}}_3 (\mathbf{p}_1 \cdot \tilde{\mathbf{p}}_2)], \end{aligned} \quad (32)$$

so that we can always reduce the operations of 6-dimensional vectors of Lorentz group to the operations of 3-dimensional vectors.

Let $(\hat{n}_1, \hat{n}_2, \hat{n}_3 = \hat{n})$ be a 3-dimensional unit vectors ($\hat{n}_i^2 = 1$) which form a right-handed orthonormal basis with $\hat{n}_1 \times \hat{n}_2 = \hat{n}_3$, and let

$$\mathbf{l}_i = \begin{pmatrix} \hat{n}_i \\ 0 \end{pmatrix}, \quad \mathbf{k}_i = \begin{pmatrix} 0 \\ \hat{n}_i \end{pmatrix} = -\tilde{\mathbf{l}}_i. \quad (33)$$

Clearly we have

$$\begin{aligned} \mathbf{l}_i \cdot \mathbf{l}_j &= \delta_{ij}, & \mathbf{l}_i \cdot \mathbf{k}_j &= 0, & \mathbf{k}_i \cdot \mathbf{k}_j &= -\delta_{ij}, \\ \mathbf{l}_i \times \mathbf{l}_j &= \epsilon_{ijk} \mathbf{l}_k, & \mathbf{l}_i \times \mathbf{k}_j &= \epsilon_{ijk} \mathbf{k}_k, \\ \mathbf{k}_i \times \mathbf{k}_j &= -\epsilon_{ijk} \mathbf{l}_k \end{aligned} \quad (34)$$

so that $(\mathbf{l}_i, \mathbf{k}_i)$, or equivalently $(\mathbf{l}_i, \tilde{\mathbf{l}}_i)$, forms an orthonormal basis of the adjoint representation of Lorentz group.

To make the desired Abelian decomposition we have to choose the gauge covariant sextet vector fields which form adjoint representation of Lorentz group which describe the desired magnetic isometry. To see what types of isometry is possible, it is important to remember that Lorentz group has two 2-dimensional maximal Abelian subgroups, A_2 whose generators are made of L_3 and K_3 and B_2 whose generators are made of $(L_1 + K_2)/\sqrt{2}$ and $(L_2 - K_1)/\sqrt{2}$ [20].

This tells that we have two possible Abelian decompositions of the gravitational connection. And in both cases the magnetic isometry is described by two, not one, commuting sextet vector fields of Lorentz group which are dual to each other. To see this let us denote one of the isometry vector field by \mathbf{p} which satisfy the isometry condition

$$D_\mu \mathbf{p} = (\partial_\mu + \mathbf{\Gamma}_\mu \times) \mathbf{p} = 0, \quad (35)$$

where we have normalized the coupling constant to be the unit (which one can always do without loss of generality). Now, notice that the above condition automatically assures

$$D_\mu \tilde{\mathbf{p}} = (\partial_\mu + \mathbf{\Gamma}_\mu \times) \tilde{\mathbf{p}} = 0, \quad (36)$$

because ϵ_{abcd} is an invariant tensor. This tells that when \mathbf{p} is an isometry, $\tilde{\mathbf{p}}$ also becomes an isometry. To verify this directly we decompose the gauge potential of Lorentz group $\mathbf{\Gamma}_\mu$ into the 3-dimensional rotation and boost parts \vec{A}_μ and \vec{B}_μ , and let

$$\mathbf{\Gamma}_\mu = \begin{pmatrix} \vec{A}_\mu \\ \vec{B}_\mu \end{pmatrix}. \quad (37)$$

With this both (35) and (36) can be written as

$$D_\mu \vec{m} = \vec{B}_\mu \times \vec{e}, \quad D_\mu \vec{e} = -\vec{B}_\mu \times \vec{m}, \quad (38)$$

where now

$$D_\mu = \partial_\mu + \vec{A}_\mu \times.$$

This confirms that (35) and (36) are actually identical to each other, which tells that the magnetic isometry in Lorentz group must be even-dimensional.

Since Lorentz group has two invariant tensors it has two Casimir invariants. And it is useful to characterize the isometry by two Casimir invariants. Let the isometry be described by \mathbf{p} and $\tilde{\mathbf{p}}$. It has two Casimir invariants α and β ,

$$\begin{aligned} \alpha &= \mathbf{p} \cdot \mathbf{p} = \vec{m}^2 - \vec{e}^2, \\ \beta &= \mathbf{p} \cdot \tilde{\mathbf{p}} = 2\vec{m} \cdot \vec{e}. \end{aligned} \quad (39)$$

But the Casimir invariants (α, β) depends on the choice of the isometry vectors. To see this consider \mathbf{p}' and $\tilde{\mathbf{p}}'$ given by a linear combination of \mathbf{p} and $\tilde{\mathbf{p}}$,

$$\mathbf{p}' = a\mathbf{p} + b\tilde{\mathbf{p}}, \quad \tilde{\mathbf{p}}' = a\tilde{\mathbf{p}} - b\mathbf{p}. \quad (40)$$

Clearly we have

$$D_\mu \mathbf{p}' = 0, \quad D_\mu \tilde{\mathbf{p}}' = 0, \quad (41)$$

so that they can also be viewed to describe the same isometry. But their Casimir invariants (α', β') are given by

$$\begin{aligned} \alpha' &= (a^2 - b^2)\alpha + 2ab\beta, \\ \beta' &= (a^2 - b^2)\beta - 2ab\alpha. \end{aligned} \quad (42)$$

And with

$$\begin{aligned} a &= \sqrt{\frac{(\alpha^2 + \beta^2)^{1/2} \pm \alpha}{2(\alpha^2 + \beta^2)}}, \\ b &= \pm \frac{|\beta|}{\beta} \sqrt{\frac{(\alpha^2 + \beta^2)^{1/2} \mp \alpha}{2(\alpha^2 + \beta^2)}}, \end{aligned}$$

we can always make

$$\alpha' = \pm 1, \quad \beta' = 0, \quad (43)$$

unless $\alpha^2 + \beta^2 = 0$. This tells that we can always choose \mathbf{p} and $\tilde{\mathbf{p}}$ in such a way to make (α, β) to be $(\pm 1, 0)$ or $(0, 0)$. Physically this means that the magnetic isometry in Einstein's theory can be classified by the non light-like (or space/time) isometry and the light-like (or null) isometry whose Casimir invariants are denoted by $(\pm 1, 0)$ and $(0, 0)$, respectively. We emphasize that once \mathbf{p} and $\tilde{\mathbf{p}}$ are chosen, (α, β) are uniquely fixed. Now we discuss the two isometries A_2 and B_2 separately.

A. A_2 (Non Light-like) Isometry

Let the maximal Abelian subgroup be A_2 . In this case the isometry is made of L_3 and K_3 , and we have two sextet vector fields which describes the isometry which are dual to each other. Let \mathbf{p} and $\tilde{\mathbf{p}}$ be the two isometry

vector fields which correspond to L_3 and K_3 . Clearly we can put

$$\mathbf{p} = f \mathbf{l}_3 = f \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{p}} = f \tilde{\mathbf{l}}_3 = f \begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix}, \quad (44)$$

where f is an arbitrary function of space-time. The Casimir invariants of the isometry vectors are given by $(f^2, 0)$. But just as in $SU(2)$ gauge theory the isometry condition (35) requires f to be a constant, because

$$\partial_\mu f^2 = \partial_\mu \mathbf{p}^2 = D_\mu \mathbf{p}^2 = 2\mathbf{p} \cdot D_\mu \mathbf{p} = 0. \quad (45)$$

And we can always normalize $f = 1$ without loss of generality.

So the A_2 isometry can always be written as

$$\mathbf{l} = \mathbf{l}_3 = \begin{pmatrix} \hat{n} \\ 0 \end{pmatrix}, \quad \tilde{\mathbf{l}} = \tilde{\mathbf{l}}_3 = \begin{pmatrix} 0 \\ -\hat{n} \end{pmatrix}, \\ D_\mu \mathbf{l} = 0, \quad D_\mu \tilde{\mathbf{l}} = 0, \quad (46)$$

whose Casimir invariants are fixed by $(1, 0)$. With this we find the restricted connection $\hat{\Gamma}_\mu$ which satisfies the isometry condition

$$\hat{\Gamma}_\mu = A_\mu \mathbf{l} - B_\mu \tilde{\mathbf{l}} - \mathbf{l} \times \partial_\mu \mathbf{l}, \\ A_\mu = \mathbf{l} \cdot \hat{\Gamma}_\mu, \quad B_\mu = \tilde{\mathbf{l}} \cdot \hat{\Gamma}_\mu, \quad (47)$$

where A_μ and B_μ are two Abelian connections of \mathbf{l} and $\tilde{\mathbf{l}}$ components which are not restricted by the isometry condition. At first glance this expression appears strange, because one expects that \mathbf{l} and $\tilde{\mathbf{l}}$ should contribute equally in the restricted connection since (35) and (36) are identical. Actually they do contribute equally because we have

$$\mathbf{l} \times \partial_\mu \mathbf{l} = -\tilde{\mathbf{l}} \times \partial_\mu \tilde{\mathbf{l}}, \quad (48)$$

so that we can express the restricted connection as

$$\hat{\Gamma}_\mu = A_\mu \mathbf{l} - B_\mu \tilde{\mathbf{l}} - \frac{1}{2}(\mathbf{l} \times \partial_\mu \mathbf{l} - \tilde{\mathbf{l}} \times \partial_\mu \tilde{\mathbf{l}}). \quad (49)$$

The restricted field strength $\hat{\mathbf{R}}_{\mu\nu}$ which describes the restricted curvature tensor $\hat{R}_{\mu\nu}^{ab}$ is given by

$$\hat{\mathbf{R}}_{\mu\nu} = \partial_\mu \hat{\Gamma}_\nu - \partial_\nu \hat{\Gamma}_\mu + \hat{\Gamma}_\mu \times \hat{\Gamma}_\nu \\ = (A_{\mu\nu} + H_{\mu\nu}) \mathbf{l} - (B_{\mu\nu} + \tilde{H}_{\mu\nu}) \tilde{\mathbf{l}}, \\ A_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \\ H_{\mu\nu} = -\mathbf{l} \cdot (\partial_\mu \mathbf{l} \times \partial_\nu \mathbf{l}), \\ \tilde{H}_{\mu\nu} = -\tilde{\mathbf{l}} \cdot (\partial_\mu \tilde{\mathbf{l}} \times \partial_\nu \tilde{\mathbf{l}}) = \tilde{\mathbf{l}} \cdot (\partial_\mu \mathbf{l} \times \partial_\nu \mathbf{l}) = 0, \\ \hat{R}_{\mu\nu}^{ab} = \hat{\mathbf{R}}_{\mu\nu} \cdot \mathbf{l}^{ab} \\ = (A_{\mu\nu} + H_{\mu\nu}) l^{ab} - B_{\mu\nu} \tilde{l}^{ab}. \quad (50)$$

Notice that $\tilde{H}_{\mu\nu}$ vanishes.

In 3-dimensional notation the isometry condition (46) can be written as

$$\hat{\Gamma}_\mu = \begin{pmatrix} \hat{A}_\mu \\ \hat{B}_\mu \end{pmatrix}, \\ \hat{D}_\mu \hat{n} = 0, \quad \hat{B}_\mu \times \hat{n} = 0, \\ \hat{D}_\mu = \partial_\mu + \hat{A}_\mu \times. \quad (51)$$

From this we have

$$\hat{A}_\mu = A_\mu \hat{n} - \hat{n} \times \partial_\mu \hat{n}, \quad \hat{B}_\mu = B_\mu \hat{n}, \\ A_\mu = \hat{n} \cdot \hat{A}_\mu, \quad B_\mu = \hat{n} \cdot \hat{B}_\mu. \quad (52)$$

Moreover, with

$$\hat{\mathbf{R}}_{\mu\nu} = \begin{pmatrix} \hat{A}_{\mu\nu} \\ \hat{B}_{\mu\nu} \end{pmatrix}, \quad (53)$$

we have

$$\hat{A}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \hat{A}_\mu \times \hat{A}_\nu \\ = (A_{\mu\nu} + H_{\mu\nu}) \hat{n} = \bar{A}_{\mu\nu} \hat{n}, \\ \hat{B}_{\mu\nu} = \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu + \hat{A}_\mu \times \hat{B}_\nu - \hat{A}_\nu \times \hat{B}_\mu \\ = \hat{D}_\mu \hat{B}_\nu - \hat{D}_\nu \hat{B}_\mu = B_{\mu\nu} \hat{n}, \\ H_{\mu\nu} = -\hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu, \\ \tilde{C}_\mu = -\hat{n}_1 \cdot \partial_\mu \hat{n}_2, \\ \bar{A}_{\mu\nu} = \partial_\mu \bar{A}_\nu - \partial_\nu \bar{A}_\mu, \quad \bar{A}_\mu = A_\mu + \tilde{C}_\mu. \quad (54)$$

Notice that \hat{A}_μ and $\hat{A}_{\mu\nu}$ are formally identical to the restricted potential and restricted field strength of $SU(2)$ gauge theory. In particular $H_{\mu\nu}$ is identical to what we have in Section II. This, together with $\tilde{H}_{\mu\nu} = 0$, tells that the topology of this isometry is identical to that of the $SU(2)$ subgroup.

With this the full connection of Lorentz group is given by

$$\Gamma_\mu = \hat{\Gamma}_\mu + \mathbf{Z}_\mu, \quad \mathbf{l} \cdot \mathbf{Z}_\mu = \tilde{\mathbf{l}} \cdot \mathbf{Z}_\mu = 0, \quad (55)$$

where \mathbf{Z}_μ is the valence connection which transforms covariantly under the Lorentz gauge transformation, or equivalently under the general coordinate transformation. The corresponding field strength $\mathbf{R}_{\mu\nu}$ which describes the curvature tensor is written as

$$\mathbf{R}_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \times \Gamma_\nu \\ = \hat{\mathbf{R}}_{\mu\nu} + \mathbf{Z}_{\mu\nu}, \\ \mathbf{Z}_{\mu\nu} = \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu + \mathbf{Z}_\mu \times \mathbf{Z}_\nu, \\ \hat{D}_\mu = \partial_\mu + \hat{\Gamma}_\mu \times, \quad (56)$$

where $\mathbf{Z}_{\mu\nu}$ is the valence part of the curvature tensor which can further be decomposed to the kinetic part $\dot{\mathbf{Z}}_{\mu\nu}$ and the potential part $\mathbf{Z}'_{\mu\nu}$,

$$\mathbf{Z}_{\mu\nu} = \dot{\mathbf{Z}}_{\mu\nu} + \mathbf{Z}'_{\mu\nu}, \\ \dot{\mathbf{Z}}_{\mu\nu} = \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu, \quad \mathbf{Z}'_{\mu\nu} = \mathbf{Z}_\mu \times \mathbf{Z}_\nu. \quad (57)$$

Now with

$$\begin{aligned}\mathbf{Z}_\mu &= Z_\mu^1 \mathbf{l}_1 - \tilde{Z}_\mu^1 \tilde{\mathbf{l}}_1 + Z_\mu^2 \mathbf{l}_2 - \tilde{Z}_\mu^2 \tilde{\mathbf{l}}_2, \\ Z_\mu^1 &= \mathbf{l}_1 \cdot \mathbf{Z}_\mu, \quad \tilde{Z}_\mu^1 = \tilde{\mathbf{l}}_1 \cdot \mathbf{Z}_\mu, \\ Z_\mu^2 &= \mathbf{l}_2 \cdot \mathbf{Z}_\mu, \quad \tilde{Z}_\mu^2 = \tilde{\mathbf{l}}_2 \cdot \mathbf{Z}_\mu,\end{aligned}\quad (58)$$

we have

$$\begin{aligned}\dot{\mathbf{Z}}_{\mu\nu} &= (\mathcal{D}_\mu Z_\nu^1 - \mathcal{D}_\nu Z_\mu^1) \mathbf{l}_1 - (\mathcal{D}_\mu \tilde{Z}_\nu^1 - \mathcal{D}_\nu \tilde{Z}_\mu^1) \tilde{\mathbf{l}}_1 \\ &\quad + (\mathcal{D}_\mu Z_\nu^2 - \mathcal{D}_\nu Z_\mu^2) \mathbf{l}_2 - (\mathcal{D}_\mu \tilde{Z}_\nu^2 - \mathcal{D}_\nu \tilde{Z}_\mu^2) \tilde{\mathbf{l}}_2, \\ \mathcal{D}_\mu Z_\nu^1 &= \partial_\mu Z_\nu^1 - \bar{A}_\mu Z_\nu^2 + B_\mu \tilde{Z}_\nu^2, \\ \mathcal{D}_\mu \tilde{Z}_\nu^1 &= \partial_\mu \tilde{Z}_\nu^1 - \bar{A}_\mu \tilde{Z}_\nu^2 - B_\mu Z_\nu^2, \\ \mathcal{D}_\mu Z_\nu^2 &= \partial_\mu Z_\nu^2 + \bar{A}_\mu Z_\nu^1 - B_\mu \tilde{Z}_\nu^1, \\ \mathcal{D}_\mu \tilde{Z}_\nu^2 &= \partial_\mu \tilde{Z}_\nu^2 + \bar{A}_\mu \tilde{Z}_\nu^1 + B_\mu Z_\nu^1, \\ \mathbf{l} \cdot \dot{\mathbf{Z}}_{\mu\nu} &= \tilde{\mathbf{l}} \cdot \dot{\mathbf{Z}}_{\mu\nu} = 0.\end{aligned}\quad (59)$$

Clearly \bar{A}_μ is identical to the dual potential we have introduced in Section II in $SU(2)$ gauge theory. Moreover, we have

$$\begin{aligned}\mathbf{Z}'_{\mu\nu} &= W_{\mu\nu} \mathbf{l} - \tilde{W}_{\mu\nu} \tilde{\mathbf{l}}, \\ W_{\mu\nu} &= \mathbf{l} \cdot (\mathbf{Z}_\mu \times \mathbf{Z}_\nu) \\ &= Z_\mu^1 Z_\nu^2 - Z_\nu^1 Z_\mu^2 - (\tilde{Z}_\mu^1 \tilde{Z}_\nu^2 - \tilde{Z}_\nu^1 \tilde{Z}_\mu^2), \\ \tilde{W}_{\mu\nu} &= \tilde{\mathbf{l}} \cdot (\mathbf{Z}_\mu \times \mathbf{Z}_\nu) \\ &= Z_\mu^1 \tilde{Z}_\nu^2 - Z_\nu^1 \tilde{Z}_\mu^2 + \tilde{Z}_\mu^1 Z_\nu^2 - \tilde{Z}_\nu^1 Z_\mu^2.\end{aligned}\quad (60)$$

With this we have the full curvature tensor

$$\begin{aligned}\mathbf{R}_{\mu\nu} &= (\bar{A}_{\mu\nu} + W_{\mu\nu}) \mathbf{l} - (B_{\mu\nu} + \tilde{W}_{\mu\nu}) \tilde{\mathbf{l}} \\ &\quad + \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu \\ &= (\mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu) \mathbf{l} - (\mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu) \tilde{\mathbf{l}} \\ &\quad + (\mathcal{D}_\mu Z_\nu^1 - \mathcal{D}_\nu Z_\mu^1) \mathbf{l}_1 - (\mathcal{D}_\mu \tilde{Z}_\nu^1 - \mathcal{D}_\nu \tilde{Z}_\mu^1) \tilde{\mathbf{l}}_1 \\ &\quad + (\mathcal{D}_\mu Z_\nu^2 - \mathcal{D}_\nu Z_\mu^2) \mathbf{l}_2 - (\mathcal{D}_\mu \tilde{Z}_\nu^2 - \mathcal{D}_\nu \tilde{Z}_\mu^2) \tilde{\mathbf{l}}_2 \\ &= R_{\mu\nu}^1 \mathbf{l}_1 - \tilde{R}_{\mu\nu}^1 \tilde{\mathbf{l}}_1 + R_{\mu\nu}^2 \mathbf{l}_2 - \tilde{R}_{\mu\nu}^2 \tilde{\mathbf{l}}_2 \\ &\quad + R_{\mu\nu} \mathbf{l} - \tilde{R}_{\mu\nu} \tilde{\mathbf{l}}, \\ \mathcal{D}_\mu \bar{A}_\nu &= \partial_\mu \bar{A}_\nu + Z_\mu^1 Z_\nu^2 - \tilde{Z}_\mu^1 \tilde{Z}_\nu^2, \\ \mathcal{D}_\mu B_\nu &= \partial_\mu B_\nu + Z_\mu^1 \tilde{Z}_\nu^2 + \tilde{Z}_\mu^1 Z_\nu^2, \\ R_{\mu\nu}^1 &= \mathcal{D}_\mu Z_\nu^1 - \mathcal{D}_\nu Z_\mu^1, \quad \tilde{R}_{\mu\nu}^1 = \mathcal{D}_\mu \tilde{Z}_\nu^1 - \mathcal{D}_\nu \tilde{Z}_\mu^1, \\ R_{\mu\nu}^2 &= \mathcal{D}_\mu Z_\nu^2 - \mathcal{D}_\nu Z_\mu^2, \quad \tilde{R}_{\mu\nu}^2 = \mathcal{D}_\mu \tilde{Z}_\nu^2 - \mathcal{D}_\nu \tilde{Z}_\mu^2, \\ R_{\mu\nu} &= \mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu = A_{\mu\nu} + H_{\mu\nu} + W_{\mu\nu}, \\ \tilde{R}_{\mu\nu} &= \mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu = B_{\mu\nu} + \tilde{W}_{\mu\nu},\end{aligned}\quad (61)$$

or equivalently

$$\begin{aligned}\mathbf{R}_{\mu\nu}^{ab} &= \mathbf{R}_{\mu\nu} \cdot \mathbf{I}^{ab} \\ &= R_{\mu\nu}^1 l_1^{ab} - \tilde{R}_{\mu\nu}^1 \tilde{l}_1^{ab} + R_{\mu\nu}^2 l_2^{ab} - \tilde{R}_{\mu\nu}^2 \tilde{l}_2^{ab} \\ &\quad + R_{\mu\nu} l^{ab} - \tilde{R}_{\mu\nu} \tilde{l}^{ab}.\end{aligned}\quad (62)$$

This is the A_2 decomposition of the curvature tensor. The similarity between this decomposition and the Abelian decomposition of $SU(2)$ is unmistakable.

To emphasize the similarity between this isometry and the $U(1)$ isometry of $SU(2)$ we introduce the complex notation

$$\begin{aligned}Z_\mu &= \frac{1}{\sqrt{2}}(Z_\mu^1 + iZ_\mu^2), \quad \tilde{Z}_\mu = \frac{1}{\sqrt{2}}(\tilde{Z}_\mu^1 + i\tilde{Z}_\mu^2), \\ \mathbf{l}_\pm &= \frac{1}{\sqrt{2}}(\mathbf{l}_1 \pm i\mathbf{l}_2), \quad \tilde{\mathbf{l}}_\pm = \frac{1}{\sqrt{2}}(\tilde{\mathbf{l}}_1 \pm i\tilde{\mathbf{l}}_2),\end{aligned}\quad (63)$$

and find

$$\begin{aligned}\dot{\mathbf{Z}}_{\mu\nu} &= (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu)^* \mathbf{l}_+ + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu) \mathbf{l}_- \\ &\quad - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu)^* \tilde{\mathbf{l}}_+ - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu) \tilde{\mathbf{l}}_-, \\ \mathcal{D}_\mu Z_\nu &= (\partial_\mu + i\bar{A}_\mu) Z_\nu - iB_\mu \tilde{Z}_\nu = \bar{D}_\mu Z_\nu - iB_\mu \tilde{Z}_\nu, \\ \mathcal{D}_\mu \tilde{Z}_\nu &= (\partial_\mu + i\bar{A}_\mu) \tilde{Z}_\nu + iB_\mu Z_\nu = \bar{D}_\mu \tilde{Z}_\nu + iB_\mu Z_\nu, \\ \bar{D}_\mu &= \partial_\mu + i\bar{A}_\mu.\end{aligned}\quad (64)$$

Here \bar{D}_μ is identical to the one we have in Section II. Moreover, the potential part of $\mathbf{Z}_{\mu\nu}$ is given by

$$\begin{aligned}\mathbf{Z}'_{\mu\nu} &= W_{\mu\nu} \mathbf{l} - \tilde{W}_{\mu\nu} \tilde{\mathbf{l}}, \\ W_{\mu\nu} &= Z_\mu^1 Z_\nu^2 - Z_\nu^1 Z_\mu^2 - (\tilde{Z}_\mu^1 \tilde{Z}_\nu^2 - \tilde{Z}_\nu^1 \tilde{Z}_\mu^2) \\ &= -i(Z_\mu^* Z_\nu - Z_\nu^* Z_\mu) + i(\tilde{Z}_\mu^* \tilde{Z}_\nu - \tilde{Z}_\nu^* \tilde{Z}_\mu), \\ \tilde{W}_{\mu\nu} &= Z_\mu^1 \tilde{Z}_\nu^2 - Z_\nu^1 \tilde{Z}_\mu^2 + \tilde{Z}_\mu^1 Z_\nu^2 - \tilde{Z}_\nu^1 Z_\mu^2 \\ &= -i(Z_\mu^* \tilde{Z}_\nu - Z_\nu^* \tilde{Z}_\mu) - i(\tilde{Z}_\mu^* Z_\nu - \tilde{Z}_\nu^* Z_\mu).\end{aligned}\quad (65)$$

With this we have

$$\begin{aligned}\mathbf{R}_{\mu\nu} &= (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu)^* \mathbf{l}_+ - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu)^* \tilde{\mathbf{l}}_+ \\ &\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu) \mathbf{l}_- - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu) \tilde{\mathbf{l}}_- \\ &\quad + (\mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu) \mathbf{l} - (\mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu) \tilde{\mathbf{l}},\end{aligned}\quad (66)$$

or

$$\begin{aligned}R_{\mu\nu}^{ab} &= (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu)^* l_+^{ab} - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu)^* \tilde{l}_+^{ab} \\ &\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu) l_-^{ab} - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu) \tilde{l}_-^{ab} \\ &\quad + (\mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu) l^{ab} - (\mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu) \tilde{l}^{ab}.\end{aligned}\quad (67)$$

This should be compared with the $SU(2)$ decomposition.

In 3-dimensional notation we have

$$\begin{aligned}\mathbf{Z}_\mu &= \begin{pmatrix} \vec{X}_\mu \\ \vec{Y}_\mu \end{pmatrix}, \\ \vec{X}_\mu &= Z_\mu^1 \hat{n}_1 + Z_\mu^2 \hat{n}_2, \quad \vec{Y}_\mu = \tilde{Z}_\mu^1 \hat{n}_1 + \tilde{Z}_\mu^2 \hat{n}_2, \\ \hat{n} \cdot \vec{X}_\mu &= 0, \quad \hat{n} \cdot \vec{Y}_\mu = 0.\end{aligned}\quad (68)$$

Moreover, with

$$\mathbf{Z}_{\mu\nu} = \begin{pmatrix} \vec{X}_{\mu\nu} \\ \vec{Y}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \dot{\vec{X}}_{\mu\nu} + \vec{X}'_{\mu\nu} \\ \dot{\vec{Y}}_{\mu\nu} + \vec{Y}'_{\mu\nu} \end{pmatrix}, \quad (69)$$

we have

$$\begin{aligned}
\dot{\vec{X}}_{\mu\nu} &= \hat{D}_\mu \vec{X}_\nu - \hat{D}_\nu \vec{X}_\mu - \vec{B}_\mu \times \vec{Y}_\nu + \vec{B}_\nu \times \vec{Y}_\mu \\
&= R_{\mu\nu}^1 \hat{n}_1 + R_{\mu\nu}^2 \hat{n}_2, \\
\dot{\vec{Y}}_{\mu\nu} &= \hat{D}_\mu \vec{Y}_\nu - \hat{D}_\nu \vec{Y}_\mu + \vec{B}_\mu \times \vec{X}_\nu - \vec{B}_\nu \times \vec{X}_\mu \\
&= \tilde{R}_{\mu\nu}^1 \hat{n}_1 + \tilde{R}_{\mu\nu}^2 \hat{n}_2, \\
\vec{X}'_{\mu\nu} &= \vec{X}_\mu \times \vec{X}_\nu - \vec{Y}_\mu \times \vec{Y}_\nu = W_{\mu\nu} \hat{n}, \\
\vec{Y}'_{\mu\nu} &= \vec{X}_\mu \times \vec{Y}_\nu + \vec{Y}_\mu \times \vec{X}_\nu = \tilde{W}_{\mu\nu} \hat{n}. \tag{70}
\end{aligned}$$

Notice that the kinetic part and the potential part of $\mathbf{Z}_{\mu\nu}$ are orthogonal to each other. Finally, with

$$\mathbf{R}_{\mu\nu} = \begin{pmatrix} \vec{A}_{\mu\nu} \\ \vec{B}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \hat{A}_{\mu\nu} + \vec{X}_{\mu\nu} \\ \hat{B}_{\mu\nu} + \vec{Y}_{\mu\nu} \end{pmatrix}, \tag{71}$$

we have

$$\begin{aligned}
\vec{A}_{\mu\nu} &= R_{\mu\nu} \hat{n} + \dot{\vec{X}}_{\mu\nu} \\
&= R_{\mu\nu}^1 \hat{n}_1 + R_{\mu\nu}^2 \hat{n}_2 + R_{\mu\nu} \hat{n}, \\
\vec{B}_{\mu\nu} &= \tilde{R}_{\mu\nu} \hat{n} + \dot{\vec{Y}}_{\mu\nu} \\
&= \tilde{R}_{\mu\nu}^1 \hat{n}_1 + \tilde{R}_{\mu\nu}^2 \hat{n}_2 + \tilde{R}_{\mu\nu} \hat{n}. \tag{72}
\end{aligned}$$

This completes the A_2 decomposition of the gravitational connection.

B. B_2 (Light-like) Isometry

This is when the isometry group is made of $(L_1 + K_2)/\sqrt{2}$ and $(L_2 - K_1)/\sqrt{2}$. Let \mathbf{p} and $\tilde{\mathbf{p}}$ be the two isometry vector fields which correspond to $(L_1 + K_2)/\sqrt{2}$ and $(L_2 - K_1)/\sqrt{2}$ which are dual to each other. In this case we can write

$$\begin{aligned}
\mathbf{p} &= f \left(\frac{\mathbf{l}_1 + \mathbf{k}_2}{\sqrt{2}} \right) = \frac{f}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}, \\
\tilde{\mathbf{p}} &= f \left(\frac{\mathbf{l}_2 - \mathbf{k}_1}{\sqrt{2}} \right) = \frac{f}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}. \tag{73}
\end{aligned}$$

But notice that the Casimir invariants (α, β) of the isometry vectors are given by $(0, 0)$ independent of f . Moreover, here (unlike the A_2 case) the isometry condition does not restrict f at all, because we have $\mathbf{p}^2 = 0$ independent of f . So the B_2 isometry vectors contain an arbitrary scalar function $f(x)$.

Let us put $f = e^\lambda$ and express the B_2 isometry by

$$\begin{aligned}
\mathbf{j} &= \frac{e^\lambda}{\sqrt{2}} (\mathbf{l}_1 + \mathbf{k}_2) = \frac{e^\lambda}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ \hat{n}_2 \end{pmatrix}, \\
\tilde{\mathbf{j}} &= \frac{e^\lambda}{\sqrt{2}} (\mathbf{l}_2 - \mathbf{k}_1) = \frac{e^\lambda}{\sqrt{2}} \begin{pmatrix} \hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}, \\
D_\mu \mathbf{j} &= 0, \quad D_\mu \tilde{\mathbf{j}} = 0, \tag{74}
\end{aligned}$$

To find the restricted connection $\hat{\Gamma}$ which satisfies the isometry condition we first introduce 4 more basis vectors in Lorentz group manifold which together with \mathbf{j} and $\tilde{\mathbf{j}}$ form a complete basis

$$\begin{aligned}
\mathbf{k} &= \frac{e^{-\lambda}}{\sqrt{2}} (\mathbf{l}_1 - \mathbf{k}_2) = \frac{e^{-\lambda}}{\sqrt{2}} \begin{pmatrix} \hat{n}_1 \\ -\hat{n}_2 \end{pmatrix}, \\
\tilde{\mathbf{k}} &= -\frac{e^{-\lambda}}{\sqrt{2}} (\mathbf{l}_2 + \mathbf{k}_1) = \frac{e^{-\lambda}}{\sqrt{2}} \begin{pmatrix} -\hat{n}_2 \\ -\hat{n}_1 \end{pmatrix}, \\
\mathbf{l} &= -\mathbf{j} \times \tilde{\mathbf{k}} = -\tilde{\mathbf{j}} \times \mathbf{k} = \begin{pmatrix} \hat{n}_3 \\ 0 \end{pmatrix}, \\
\tilde{\mathbf{l}} &= \mathbf{j} \times \mathbf{k} = -\tilde{\mathbf{j}} \times \tilde{\mathbf{k}} = \begin{pmatrix} 0 \\ -\hat{n}_3 \end{pmatrix}. \tag{75}
\end{aligned}$$

Notice that 4 of them are null vectors,

$$\mathbf{j}^2 = \tilde{\mathbf{j}}^2 = \mathbf{k}^2 = \tilde{\mathbf{k}}^2 = 0, \tag{76}$$

but we have

$$\mathbf{j} \cdot \mathbf{k} = -\tilde{\mathbf{j}} \cdot \tilde{\mathbf{k}} = 1, \quad \mathbf{l}^2 = -\tilde{\mathbf{l}}^2 = 1. \tag{77}$$

All other scalar products of the basis vectors vanish. Moreover we have

$$\begin{aligned}
\mathbf{j} \times \mathbf{l} &= -\tilde{\mathbf{j}} \times \tilde{\mathbf{l}} = -\tilde{\mathbf{j}}, \quad \tilde{\mathbf{j}} \times \mathbf{l} = \mathbf{j} \times \tilde{\mathbf{l}} = \mathbf{j}, \\
\mathbf{k} \times \mathbf{l} &= -\tilde{\mathbf{k}} \times \tilde{\mathbf{l}} = \tilde{\mathbf{k}}, \quad \tilde{\mathbf{k}} \times \mathbf{l} = \mathbf{k} \times \tilde{\mathbf{l}} = -\mathbf{k}. \tag{78}
\end{aligned}$$

From this we find the following restricted connection for the B_2 isometry,

$$\begin{aligned}
\hat{\Gamma}_\mu &= \Gamma_\mu \mathbf{j} - \tilde{\Gamma}_\mu \tilde{\mathbf{j}} - \mathbf{k} \times \partial_\mu \mathbf{j} \\
&= \Gamma_\mu \mathbf{j} - \tilde{\Gamma}_\mu \tilde{\mathbf{j}} - \frac{1}{2} (\mathbf{k} \times \partial_\mu \mathbf{j} - \tilde{\mathbf{k}} \times \partial_\mu \tilde{\mathbf{j}}), \\
\Gamma_\mu &= \mathbf{k} \cdot \Gamma_\mu, \quad \tilde{\Gamma}_\mu = \tilde{\mathbf{k}} \cdot \Gamma_\mu, \\
\mathbf{k} \times \partial_\mu \mathbf{j} &= -\tilde{\mathbf{k}} \times \partial_\mu \tilde{\mathbf{j}}, \tag{79}
\end{aligned}$$

where Γ_μ and $\tilde{\Gamma}_\mu$ are two Abelian connections of \mathbf{j} and $\tilde{\mathbf{j}}$ components which are not restricted by the isometry condition.

The restricted curvature tensor $\hat{\mathbf{R}}_{\mu\nu}$ is given by

$$\begin{aligned}
\hat{\mathbf{R}}_{\mu\nu} &= \partial_\mu \hat{\Gamma}_\nu - \partial_\nu \hat{\Gamma}_\mu + \hat{\Gamma}_\mu \times \hat{\Gamma}_\nu \\
&= (\Gamma_{\mu\nu} + H_{\mu\nu}) \mathbf{j} - (\tilde{\Gamma}_{\mu\nu} + \tilde{H}_{\mu\nu}) \tilde{\mathbf{j}}, \\
\Gamma_{\mu\nu} &= \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu, \quad \tilde{\Gamma}_{\mu\nu} = \partial_\mu \tilde{\Gamma}_\nu - \partial_\nu \tilde{\Gamma}_\mu, \\
H_{\mu\nu} &= -\mathbf{k} \cdot (\partial_\mu \mathbf{j} \times \partial_\nu \mathbf{k} - \partial_\nu \mathbf{j} \times \partial_\mu \mathbf{k}), \\
\tilde{H}_{\mu\nu} &= -\tilde{\mathbf{k}} \cdot (\partial_\mu \tilde{\mathbf{j}} \times \partial_\nu \tilde{\mathbf{k}} - \partial_\nu \tilde{\mathbf{j}} \times \partial_\mu \tilde{\mathbf{k}}), \tag{80}
\end{aligned}$$

so that

$$\hat{\mathbf{R}}_{\mu\nu}^{ab} = (\Gamma_{\mu\nu} + H_{\mu\nu}) j^{ab} - (\tilde{\Gamma}_{\mu\nu} + \tilde{H}_{\mu\nu}) \tilde{j}^{ab}. \tag{81}$$

Notice that $\hat{\mathbf{R}}_{\mu\nu}$ is orthogonal to \mathbf{l} and $\tilde{\mathbf{l}}$. This should be contrasted with the restricted curvature tensor (50) of the A_2 isometry.

In 3-dimensional notation the isometry condition (74) is written as

$$\begin{aligned}\hat{\Gamma}_\mu &= \begin{pmatrix} \hat{A}_\mu \\ \hat{B}_\mu \end{pmatrix}, \\ \hat{D}_\mu \hat{n}_1 &= \hat{B}_\mu \times \hat{n}_2 - (\partial_\mu \lambda) \hat{n}_1, \\ \hat{D}_\mu \hat{n}_2 &= -\hat{B}_\mu \times \hat{n}_1 - (\partial_\mu \lambda) \hat{n}_2.\end{aligned}\quad (82)$$

From this we have

$$\begin{aligned}\hat{A}_\mu &= A_\mu^1 \hat{n}_1 + A_\mu^2 \hat{n}_2 + (\hat{n}_1 \cdot \partial_\mu \hat{n}_2) \hat{n}_3 \\ &= \left(\frac{e^\lambda}{\sqrt{2}} \Gamma_\mu + \frac{\hat{n}_2 \cdot \partial_\mu \hat{n}_3}{2} \right) \hat{n}_1 - \left(\frac{e^\lambda}{\sqrt{2}} \tilde{\Gamma}_\mu - \frac{\hat{n}_3 \cdot \partial_\mu \hat{n}_1}{2} \right) \hat{n}_2 \\ &\quad + (\hat{n}_1 \cdot \partial_\mu \hat{n}_2) \hat{n}_3, \\ \hat{B}_\mu &= B_\mu^1 \hat{n}_1 + B_\mu^2 \hat{n}_2 - (\partial_\mu \lambda) \hat{n}_3 \\ &= \left(\frac{e^\lambda}{\sqrt{2}} \tilde{\Gamma}_\mu + \frac{\hat{n}_3 \cdot \partial_\mu \hat{n}_1}{2} \right) \hat{n}_1 + \left(\frac{e^\lambda}{\sqrt{2}} \Gamma_\mu - \frac{\hat{n}_2 \cdot \partial_\mu \hat{n}_3}{2} \right) \hat{n}_2 \\ &\quad - (\partial_\mu \lambda) \hat{n}_3, \\ A_\mu^1 &= \frac{e^\lambda}{\sqrt{2}} (\Gamma_\mu - \tilde{C}_\mu^1), \quad A_\mu^2 = -\frac{e^\lambda}{\sqrt{2}} (\tilde{\Gamma}_\mu - \tilde{C}_\mu^2), \\ B_\mu^1 &= \frac{e^\lambda}{\sqrt{2}} (\tilde{\Gamma}_\mu + \tilde{C}_\mu^2), \quad B_\mu^2 = \frac{e^\lambda}{\sqrt{2}} (\Gamma_\mu + \tilde{C}_\mu^1), \\ \tilde{C}_\mu^1 &= -\frac{e^{-\lambda}}{\sqrt{2}} \hat{n}_2 \cdot \partial_\mu \hat{n}_3, \\ \tilde{C}_\mu^2 &= -\frac{e^{-\lambda}}{\sqrt{2}} \hat{n}_1 \cdot \partial_\mu \hat{n}_3,\end{aligned}\quad (83)$$

so that

$$\begin{aligned}\hat{A}_\mu &= -\hat{n}_3 \times \hat{B}_\mu + \frac{1}{2} \epsilon_{ijk} (\hat{n}_i \cdot \partial_\mu \hat{n}_j) \hat{n}_k \\ &= B_\mu^2 \hat{n}_1 - B_\mu^1 \hat{n}_2 + \frac{1}{2} \epsilon_{ijk} (\hat{n}_i \cdot \partial_\mu \hat{n}_j) \hat{n}_k, \\ \hat{B}_\mu &= \hat{n}_3 \times \hat{A}_\mu - \partial_\mu \hat{n}_3 - (\partial_\mu \lambda) \hat{n}_3 \\ &= -A_\mu^2 \hat{n}_1 + A_\mu^1 \hat{n}_2 - \partial_\mu \hat{n}_3 - (\partial_\mu \lambda) \hat{n}_3.\end{aligned}\quad (84)$$

Notice that both \hat{A}_μ and \hat{B}_μ have non-vanishing \hat{n}_3 components.

With

$$\hat{\mathbf{R}}_{\mu\nu} = \begin{pmatrix} \hat{A}_{\mu\nu} \\ \hat{B}_{\mu\nu} \end{pmatrix}$$

we have

$$\begin{aligned}\hat{A}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \hat{A}_\mu \times \hat{A}_\nu - \hat{B}_\mu \times \hat{B}_\nu \\ &= \frac{e^\lambda}{\sqrt{2}} (\Gamma_{\mu\nu} + H_{\mu\nu}) \hat{n}_1 - \frac{e^\lambda}{\sqrt{2}} (\tilde{\Gamma}_{\mu\nu} + \tilde{H}_{\mu\nu}) \hat{n}_2 \\ &\quad = A_{\mu\nu}^1 \hat{n}_1 + A_{\mu\nu}^2 \hat{n}_2, \\ \hat{B}_{\mu\nu} &= \partial_\mu \hat{B}_\nu - \partial_\nu \hat{B}_\mu + \hat{A}_\mu \times \hat{B}_\nu - \hat{A}_\nu \times \hat{B}_\mu \\ &\quad = \hat{D}_\mu \hat{B}_\nu - \hat{D}_\nu \hat{B}_\mu \\ &= \frac{e^\lambda}{\sqrt{2}} (\tilde{\Gamma}_{\mu\nu} + \tilde{H}_{\mu\nu}) \hat{n}_1 + \frac{e^\lambda}{\sqrt{2}} (\Gamma_{\mu\nu} + H_{\mu\nu}) \hat{n}_2 \\ &\quad = B_{\mu\nu}^1 \hat{n}_1 + B_{\mu\nu}^2 \hat{n}_2,\end{aligned}\quad (85)$$

where

$$\begin{aligned}H_{\mu\nu} &= \partial_\mu \tilde{C}_\nu^1 - \partial_\nu \tilde{C}_\mu^1 = \frac{e^{-\lambda}}{\sqrt{2}} \left(-\hat{n}_1 \cdot (\partial_\mu \hat{n}_1 \times \partial_\nu \hat{n}_1) \right. \\ &\quad \left. + \hat{n}_2 \cdot (\partial_\mu \lambda \partial_\nu \hat{n}_3 - \partial_\nu \lambda \partial_\mu \hat{n}_3) \right), \\ \tilde{H}_{\mu\nu} &= \partial_\mu \tilde{C}_\nu^2 - \partial_\nu \tilde{C}_\mu^2 = \frac{e^{-\lambda}}{\sqrt{2}} \left(\hat{n}_2 \cdot (\partial_\mu \hat{n}_2 \times \partial_\nu \hat{n}_2) \right. \\ &\quad \left. + \hat{n}_1 \cdot (\partial_\mu \lambda \partial_\nu \hat{n}_3 - \partial_\nu \lambda \partial_\mu \hat{n}_3) \right), \\ A_{\mu\nu}^1 &= B_{\mu\nu}^2 = \frac{e^\lambda}{\sqrt{2}} (\partial_\mu K_\nu - \partial_\nu K_\mu), \\ A_{\mu\nu}^2 &= -B_{\mu\nu}^1 = -\frac{e^\lambda}{\sqrt{2}} (\partial_\mu \tilde{K}_\nu - \partial_\nu \tilde{K}_\mu), \\ K_\mu &= \Gamma_\mu + \tilde{C}_\mu^1, \quad \tilde{K}_\mu = \tilde{\Gamma}_\mu + \tilde{C}_\mu^2,\end{aligned}\quad (86)$$

so that

$$\hat{A}_{\mu\nu} = -\hat{n}_3 \times \hat{B}_{\mu\nu}, \quad \hat{B}_{\mu\nu} = \hat{n}_3 \times \hat{A}_{\mu\nu}. \quad (87)$$

Notice that both $\hat{A}_{\mu\nu}$ and $\hat{B}_{\mu\nu}$ are orthogonal to \hat{n}_3 , although \hat{A}_μ and \hat{B}_μ are not.

With this we obtain the full gauge potential of Lorentz group by adding the valence connection \mathbf{Z}_μ ,

$$\begin{aligned}\Gamma_\mu &= \hat{\Gamma}_\mu + \mathbf{Z}_\mu, \\ \mathbf{k} \cdot \mathbf{Z}_\mu &= \tilde{\mathbf{k}} \cdot \mathbf{Z}_\mu = 0.\end{aligned}\quad (88)$$

With

$$\begin{aligned}\mathbf{Z}_\mu &= J_\mu \mathbf{k} - \tilde{J}_\mu \tilde{\mathbf{k}} + L_\mu \mathbf{l} - \tilde{L}_\mu \tilde{\mathbf{l}}, \\ J_\mu &= \mathbf{j} \cdot \mathbf{Z}_\mu, \quad \tilde{J}_\mu = \tilde{\mathbf{j}} \cdot \mathbf{Z}_\mu, \\ L_\mu &= \mathbf{l} \cdot \mathbf{Z}_\mu, \quad \tilde{L}_\mu = \tilde{\mathbf{l}} \cdot \mathbf{Z}_\mu,\end{aligned}\quad (89)$$

we have

$$\begin{aligned}\dot{\mathbf{Z}}_{\mu\nu} &= \hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu \\ &= U_{\mu\nu} \mathbf{j} - \tilde{U}_{\mu\nu} \tilde{\mathbf{j}} + (\partial_\mu J_\nu - \partial_\nu J_\mu) \mathbf{k} - (\partial_\mu \tilde{J}_\nu - \partial_\nu \tilde{J}_\mu) \tilde{\mathbf{k}} \\ &\quad + (\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu) \mathbf{l} - (\mathcal{D}_\mu \tilde{L}_\nu - \mathcal{D}_\nu \tilde{L}_\mu) \tilde{\mathbf{l}}, \\ U_{\mu\nu} &= -K_\mu \tilde{L}_\nu - \tilde{K}_\mu L_\nu + (K_\nu \tilde{L}_\mu + \tilde{K}_\nu L_\mu), \\ \tilde{U}_{\mu\nu} &= K_\mu L_\nu - \tilde{K}_\mu \tilde{L}_\nu - (K_\nu L_\mu - \tilde{K}_\nu \tilde{L}_\mu), \\ \mathcal{D}_\mu L_\nu &= \partial_\mu L_\nu + K_\mu \tilde{J}_\nu + \tilde{K}_\mu J_\nu, \\ \mathcal{D}_\mu \tilde{L}_\nu &= \partial_\mu \tilde{L}_\nu - K_\mu J_\nu + \tilde{K}_\mu \tilde{J}_\nu, \\ \mathbf{Z}'_{\mu\nu} &= \mathbf{Z}_\mu \times \mathbf{Z}_\nu = V_{\mu\nu} \mathbf{k} - \tilde{V}_{\mu\nu} \tilde{\mathbf{k}}, \\ V_{\mu\nu} &= J_\mu \tilde{L}_\nu + \tilde{J}_\mu L_\nu - (J_\nu \tilde{L}_\mu + \tilde{J}_\nu L_\mu), \\ \tilde{V}_{\mu\nu} &= \tilde{J}_\mu \tilde{L}_\nu - J_\mu L_\nu - (\tilde{J}_\nu \tilde{L}_\mu - J_\nu L_\mu),\end{aligned}\quad (90)$$

so that

$$\begin{aligned}\mathbf{Z}_{\mu\nu} &= \dot{\mathbf{Z}}_{\mu\nu} + \mathbf{Z}'_{\mu\nu} = U_{\mu\nu} \mathbf{j} - \tilde{U}_{\mu\nu} \tilde{\mathbf{j}} \\ &\quad + (\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu) \mathbf{k} - (\mathcal{D}_\mu \tilde{J}_\nu - \mathcal{D}_\nu \tilde{J}_\mu) \tilde{\mathbf{k}} \\ &\quad + (\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu) \mathbf{l} - (\mathcal{D}_\mu \tilde{L}_\nu - \mathcal{D}_\nu \tilde{L}_\mu) \tilde{\mathbf{l}}, \\ \mathcal{D}_\mu J_\nu &= \partial_\mu J_\nu - \tilde{L}_\mu J_\nu - L_\mu \tilde{J}_\nu, \\ \mathcal{D}_\mu \tilde{J}_\nu &= \partial_\mu \tilde{J}_\nu - \tilde{L}_\mu \tilde{J}_\nu + L_\mu J_\nu.\end{aligned}\quad (91)$$

Notice that in this case the kinetic part $\dot{\mathbf{Z}}_{\mu\nu}$ contains all six components, but the potential part $\mathbf{Z}'_{\mu\nu}$ has only \mathbf{k} and $\tilde{\mathbf{k}}$ components. With this we have the full curvature tensor

$$\begin{aligned}
\mathbf{R}_{\mu\nu} &= \hat{\mathbf{R}}_{\mu\nu} + \dot{\mathbf{Z}}_{\mu\nu} + \mathbf{Z}'_{\mu\nu} \\
&= (\Gamma_{\mu\nu} + H_{\mu\nu} + U_{\mu\nu})\mathbf{j} - (\tilde{\Gamma}_{\mu\nu} + \tilde{H}_{\mu\nu} + \tilde{U}_{\mu\nu})\tilde{\mathbf{j}} \\
&\quad + (\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu)\mathbf{k} - (\mathcal{D}_\mu \tilde{J}_\nu - \mathcal{D}_\nu \tilde{J}_\mu)\tilde{\mathbf{k}} \\
&\quad + (\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu)\mathbf{l} - (\mathcal{D}_\mu \tilde{L}_\nu - \mathcal{D}_\nu \tilde{L}_\mu)\tilde{\mathbf{l}} \\
&= (\mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu)\mathbf{j} - (\mathcal{D}_\mu \tilde{K}_\nu - \mathcal{D}_\nu \tilde{K}_\mu)\tilde{\mathbf{j}} \\
&\quad + (\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu)\mathbf{k} - (\mathcal{D}_\mu \tilde{J}_\nu - \mathcal{D}_\nu \tilde{J}_\mu)\tilde{\mathbf{k}} \\
&\quad + (\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu)\mathbf{l} - (\mathcal{D}_\mu \tilde{L}_\nu - \mathcal{D}_\nu \tilde{L}_\mu)\tilde{\mathbf{l}} \\
&= K_{\mu\nu}\mathbf{j} - \tilde{K}_{\mu\nu}\tilde{\mathbf{j}} + J_{\mu\nu}\mathbf{k} - \tilde{J}_{\mu\nu}\tilde{\mathbf{k}} + L_{\mu\nu}\mathbf{l} - \tilde{L}_{\mu\nu}\tilde{\mathbf{l}}, \\
&\quad \mathcal{D}_\mu K_\nu = \partial_\mu K_\nu + \tilde{L}_\mu K_\nu + L_\mu \tilde{K}_\nu, \\
&\quad \mathcal{D}_\mu \tilde{K}_\nu = \partial_\mu \tilde{K}_\nu + \tilde{L}_\mu \tilde{K}_\nu - L_\mu K_\nu, \\
&\quad K_{\mu\nu} = \Gamma_{\mu\nu} + H_{\mu\nu} + U_{\mu\nu} = \mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu, \\
&\quad \tilde{K}_{\mu\nu} = \tilde{\Gamma}_{\mu\nu} + \tilde{H}_{\mu\nu} + \tilde{U}_{\mu\nu} = \mathcal{D}_\mu \tilde{K}_\nu - \mathcal{D}_\nu \tilde{K}_\mu, \\
&\quad J_{\mu\nu} = \mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu, \quad \tilde{J}_{\mu\nu} = \mathcal{D}_\mu \tilde{J}_\nu - \mathcal{D}_\nu \tilde{J}_\mu, \\
&\quad L_{\mu\nu} = \mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu, \quad \tilde{L}_{\mu\nu} = \mathcal{D}_\mu \tilde{L}_\nu - \mathcal{D}_\nu \tilde{L}_\mu, \quad (92)
\end{aligned}$$

or equivalently

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= \mathbf{R}_{\mu\nu} \cdot \mathbf{I}^{ab} \\
&= K_{\mu\nu}j^{ab} - \tilde{K}_{\mu\nu}\tilde{j}^{ab} + J_{\mu\nu}k^{ab} - \tilde{J}_{\mu\nu}\tilde{k}^{ab} \\
&\quad + L_{\mu\nu}l^{ab} - \tilde{L}_{\mu\nu}\tilde{l}^{ab}, \quad (93)
\end{aligned}$$

This is the B_2 decomposition of the curvature tensor.

With complex notation

$$\begin{aligned}
\mathbf{k}_\pm &= \frac{1}{\sqrt{2}}(\mathbf{k} \pm i\mathbf{l}), \quad \tilde{\mathbf{k}}_\pm = \frac{1}{\sqrt{2}}(\tilde{\mathbf{k}} \pm i\tilde{\mathbf{l}}), \\
Z_\mu &= \frac{1}{\sqrt{2}}(J_\mu + iL_\mu), \quad \tilde{Z}_\mu = \frac{1}{\sqrt{2}}(\tilde{J}_\mu + i\tilde{L}_\mu), \\
Z'_\mu &= \frac{1}{\sqrt{2}}(K_\mu + iL_\mu) = Z_\mu - \frac{1}{\sqrt{2}}B_\mu^-, \\
\tilde{Z}'_\mu &= \frac{1}{\sqrt{2}}(\tilde{K}_\mu + i\tilde{L}_\mu) = \tilde{Z}_\mu - \frac{1}{\sqrt{2}}\tilde{B}_\mu^-, \\
B_\mu^\pm &= J_\mu \pm K_\mu, \quad \tilde{B}_\mu^\pm = \tilde{J}_\mu \pm \tilde{K}_\mu, \quad (94)
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathbf{Z}_{\mu\nu} &= i(\tilde{Z}'_\mu Z'_\nu - \tilde{Z}'_\nu Z'_\mu + Z'_\mu \tilde{Z}'_\nu - Z'_\nu \tilde{Z}'_\mu)\mathbf{j} \\
&\quad + i(\tilde{Z}'_\mu \tilde{Z}'_\nu - \tilde{Z}'_\nu \tilde{Z}'_\mu + Z'_\mu Z'_\nu - Z'_\nu Z'_\mu)\tilde{\mathbf{j}} \\
&\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu)^* \mathbf{k}_+ - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu)^* \tilde{\mathbf{k}}_+, \\
&\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu) \mathbf{k}_- - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu) \tilde{\mathbf{k}}_-, \\
&\quad \mathcal{D}_\mu Z_\nu = \partial_\mu Z_\nu - \frac{i}{2}(\tilde{B}_\mu^- Z_\nu - \tilde{B}_\mu^+ Z_\nu^* \\
&\quad \quad + B_\mu^- \tilde{Z}_\nu - B_\mu^+ \tilde{Z}_\nu^*), \\
&\quad \mathcal{D}_\mu \tilde{Z}_\nu = \partial_\mu \tilde{Z}_\nu - \frac{i}{2}(\tilde{B}_\mu^- \tilde{Z}_\nu - \tilde{B}_\mu^+ \tilde{Z}_\nu^* \\
&\quad \quad - B_\mu^- Z_\nu + B_\mu^+ Z_\nu^*). \quad (95)
\end{aligned}$$

With this we have

$$\begin{aligned}
\mathbf{R}_{\mu\nu} &= (\mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu) \mathbf{j} - (\mathcal{D}_\mu \tilde{K}_\nu - \mathcal{D}_\nu \tilde{K}_\mu) \tilde{\mathbf{j}} \\
&\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu)^* \mathbf{k}_+ - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu)^* \tilde{\mathbf{k}}_+ \\
&\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu) \mathbf{k}_- \\
&\quad - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu) \tilde{\mathbf{k}}_-, \quad (96)
\end{aligned}$$

or

$$\begin{aligned}
R_{\mu\nu}{}^{ab} &= (\mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu) j^{ab} - (\mathcal{D}_\mu \tilde{K}_\nu - \mathcal{D}_\nu \tilde{K}_\mu) \tilde{j}^{ab} \\
&\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu)^* k_+^{ab} - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu)^* \tilde{k}_+^{ab} \\
&\quad + (\mathcal{D}_\mu Z_\nu - \mathcal{D}_\nu Z_\mu) k_-^{ab} \\
&\quad - (\mathcal{D}_\mu \tilde{Z}_\nu - \mathcal{D}_\nu \tilde{Z}_\mu) \tilde{k}_-^{ab}. \quad (97)
\end{aligned}$$

This should be compared with the A_2 result (66) or (67).

In 3-dimensional notation, we have

$$\begin{aligned}
\mathbf{Z}_\mu &= \begin{pmatrix} \vec{X}_\mu \\ \vec{Y}_\mu \end{pmatrix}, \\
\vec{X}_\mu &= \frac{e^{-\lambda}}{\sqrt{2}}(J_\mu \hat{n}_1 + \tilde{J}_\mu \hat{n}_2) + L_\mu \hat{n}_3, \\
\vec{Y}_\mu &= \frac{e^{-\lambda}}{\sqrt{2}}(\tilde{J}_\mu \hat{n}_1 - J_\mu \hat{n}_2) + \tilde{L}_\mu \hat{n}_3, \quad (98)
\end{aligned}$$

so that

$$\begin{aligned}
\hat{n}_1 \cdot \vec{X}_\mu + \hat{n}_2 \cdot \vec{Y}_\mu &= 0, \\
\hat{n}_2 \cdot \vec{X}_\mu - \hat{n}_1 \cdot \vec{Y}_\mu &= 0, \\
\hat{n}_3 \times \vec{Y}_\mu &= -\hat{n}_3 \times (\hat{n}_3 \times \vec{X}_\mu). \quad (99)
\end{aligned}$$

Moreover, with

$$\mathbf{Z}_{\mu\nu} = \begin{pmatrix} \vec{X}_{\mu\nu} \\ \vec{Y}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \dot{\vec{X}}_{\mu\nu} + \vec{X}'_{\mu\nu} \\ \dot{\vec{Y}}_{\mu\nu} + \vec{Y}'_{\mu\nu} \end{pmatrix}, \quad (100)$$

we have

$$\begin{aligned}
\dot{\vec{X}}_{\mu\nu} &= \left\{ \frac{e^\lambda}{\sqrt{2}} U_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu J_\nu - \partial_\nu J_\mu) \right\} \hat{n}_1 \\
&\quad - \left\{ \frac{e^\lambda}{\sqrt{2}} \tilde{U}_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu \tilde{J}_\nu - \partial_\nu \tilde{J}_\mu) \right\} \hat{n}_2 + L_{\mu\nu} \hat{n}_3, \\
\dot{\vec{Y}}_{\mu\nu} &= \left\{ \frac{e^\lambda}{\sqrt{2}} \tilde{U}_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu \tilde{J}_\nu - \partial_\nu \tilde{J}_\mu) \right\} \hat{n}_1 \\
&\quad + \left\{ \frac{e^\lambda}{\sqrt{2}} U_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}} (\partial_\mu J_\nu - \partial_\nu J_\mu) \right\} \hat{n}_2 + \tilde{L}_{\mu\nu} \hat{n}_3, \\
\vec{X}'_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}} (V_{\mu\nu} \hat{n}_1 + \tilde{V}_{\mu\nu} \hat{n}_2), \\
\vec{Y}'_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}} (\tilde{V}_{\mu\nu} \hat{n}_1 - V_{\mu\nu} \hat{n}_2), \quad (101)
\end{aligned}$$

so that

$$\vec{X}_{\mu\nu} = \left(\frac{e^\lambda}{\sqrt{2}} U_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}} J_{\mu\nu} \right) \hat{n}_1$$

$$\begin{aligned}
& -\left(\frac{e^\lambda}{\sqrt{2}}\tilde{U}_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}\tilde{J}_{\mu\nu}\right)\hat{n}_2 + L_{\mu\nu}\hat{n}_3, \\
& \tilde{Y}_{\mu\nu} = \left(\frac{e^\lambda}{\sqrt{2}}\tilde{U}_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}}\tilde{J}_{\mu\nu}\right)\hat{n}_1 \\
& + \left(\frac{e^\lambda}{\sqrt{2}}U_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}J_{\mu\nu}\right)\hat{n}_2 + \tilde{L}_{\mu\nu}\hat{n}_3. \quad (102)
\end{aligned}$$

Finally with

$$\mathbf{R}_{\mu\nu} = \begin{pmatrix} \vec{A}_{\mu\nu} \\ \vec{B}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} \hat{A}_{\mu\nu} + \vec{X}_{\mu\nu} \\ \hat{B}_{\mu\nu} + \vec{Y}_{\mu\nu} \end{pmatrix}, \quad (103)$$

we have

$$\begin{aligned}
& \vec{A}_{\mu\nu} = \left(\frac{e^\lambda}{\sqrt{2}}K_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}}J_{\mu\nu}\right)\hat{n}_1 \\
& - \left(\frac{e^\lambda}{\sqrt{2}}\tilde{K}_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}\tilde{J}_{\mu\nu}\right)\hat{n}_2 + L_{\mu\nu}\hat{n}_3, \\
& \vec{B}_{\mu\nu} = \left(\frac{e^\lambda}{\sqrt{2}}\tilde{K}_{\mu\nu} + \frac{e^{-\lambda}}{\sqrt{2}}\tilde{J}_{\mu\nu}\right)\hat{n}_1 \\
& + \left(\frac{e^\lambda}{\sqrt{2}}K_{\mu\nu} - \frac{e^{-\lambda}}{\sqrt{2}}J_{\mu\nu}\right)\hat{n}_2 + \tilde{L}_{\mu\nu}\hat{n}_3. \quad (104)
\end{aligned}$$

This completes the B_2 decomposition of the gravitational connection.

The above result tells that there exist two different Abelian decompositions of the gravitational connection and the curvature tensor which decompose them into the restricted part and the valence part. This allows us to decompose the Einstein's theory in terms of the restricted part and the valence part.

IV. ABELIAN DECOMPOSITION OF EINSTEIN'S THEORY

Now we are ready to discuss the decomposition of Einstein's theory. Since the Einstein-Hilbert action is described by the metric we have to express the above decomposition of the gravitational connection in terms of the metric. To do this we use the first order formalism of Einstein theory. In the absence of the matter field, the Einstein-Hilbert action in the first order formalism is given by

$$\begin{aligned}
S[e_a^\mu, \Gamma_\mu] &= \frac{1}{16\pi G_N} \int e \left(e_a^\mu e_b^\nu \mathbf{I}^{ab} \cdot \mathbf{R}_{\mu\nu} \right) d^4x \\
&= \frac{1}{16\pi G_N} \int e \left(\mathbf{g}_{\mu\nu} \cdot \mathbf{R}^{\mu\nu} \right) d^4x, \\
e &= \text{Det}(e_{\mu a}), \quad \mathbf{g}_{\mu\nu} = e_\mu^a e_\nu^b \mathbf{I}_{ab}, \\
g_{\mu\nu}^{ab} &= (e_\mu^a e_\nu^b - e_\nu^a e_\mu^b) = g_{[\mu\nu]}^{[ab]}. \quad (105)
\end{aligned}$$

Here we have introduced the Lorentz covariant four index metric tensor $\mathbf{g}_{\mu\nu}$ (which should not be confused with the

two index space-time metric $g_{\mu\nu}$) which forms an adjoint representation of Lorentz group. Notice that $\mathbf{g}_{\mu\nu}$ is antisymmetric in μ and ν . Clearly this Lorentz covariant metric becomes the natural metric which plays the role of $g_{\mu\nu}$ in this gauge formalism.

From (105) we have the following equation of motion

$$\begin{aligned}
& \delta e_{\mu a}; \quad \mathbf{g}_{\mu\nu} \cdot \mathbf{R}^{\nu\rho} e_{\rho a} = R_{\mu a} = 0 \\
& \delta \Gamma_\mu; \quad \mathcal{D}_\mu \mathbf{g}^{\mu\nu} = (\nabla_\mu + \Gamma_\mu \times) \mathbf{g}^{\mu\nu} = 0, \quad (106)
\end{aligned}$$

where $R_{\mu a} = e^{\nu b} R_{\mu\nu ab}$ is the Ricci tensor and \mathcal{D}_μ is generally and gauge covariant derivative. Clearly the first equation is nothing but the Einstein's equation in the absence of matter field. But the Ricci tensor is written in terms of the gauge potential, not the metric.

To understand the meaning of the second equation, notice that the second equation tells that $\mathbf{g}_{\mu\nu}$ is invariant under the parallel transport along the ∂_μ -direction defined by the gauge potential Γ_μ , which puts a strong constraint on Γ_μ . In fact from this one can show that Γ_μ is given by

$$\begin{aligned}
\Gamma_\mu \cdot \mathbf{I}^{ab} &= \frac{1}{2} (e^{a\nu} e_{c\mu} \partial^b e^c_\nu + e^{a\nu} \partial_\mu e^b_\nu + \partial^b e^a_\mu \\
& - e^{b\nu} e_{c\mu} \partial^a e^c_\nu - e^{b\nu} \partial_\mu e^a_\nu - \partial^a e^b_\mu) = \Gamma_\mu^{ab}. \quad (107)
\end{aligned}$$

But this, of course, is the Levi-Civita connection written in the tetrad basis. This confirms that the gauge potential Γ_μ of Lorentz group becomes the (torsion-free) spin connection ω_μ^{ab} , which assures that (106) indeed describes the Einstein's general relativity. But remember that in general it can have torsion when a spinor source is present [7].

This tells that the second equation of (106) is nothing but the metric-compatibility condition of the connection

$$\mathcal{D}_\mu \mathbf{g}^{\mu\nu} = 0 \iff \nabla_\alpha g_{\mu\nu} = 0. \quad (108)$$

But actually in the Lorentz gauge formalism of Einstein's theory we have this metric-compatibility from the beginning, because we already have

$$D_\mu \eta_{ab} = 0. \quad (109)$$

Indeed, with this and with the identity

$$\mathcal{D}_\mu e_\nu^a = \partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\alpha e_\alpha^a + \Gamma_\mu^a{}_\nu e_\nu^b = 0, \quad (110)$$

$\mathcal{D}_\mu \mathbf{g}^{\mu\nu}$ is reduced to

$$D_\mu \mathbf{I}^{ab} = 0, \quad (111)$$

which becomes an identity with (109). So the second equation of (106) can actually be viewed as an identity.

A. A_2 (Non Light-like) Decomposition

With this preliminary, we discuss the decomposition of Einstein's theory with the A_2 isometry (the space/time

isometry) first. For this we introduce two projection operators which project out the isometry components,

$$\begin{aligned}\Sigma_{ab} &= l_{ab} \mathbf{1} - \tilde{l}_{ab} \tilde{\mathbf{1}}, \\ \Pi_{ab} &= \mathbf{I}_{ab} - \Sigma_{ab} = l_{ab}^1 \mathbf{l}_1 - \tilde{l}_{ab}^1 \tilde{\mathbf{l}}_1 + l_{ab}^2 \mathbf{l}_2 - \tilde{l}_{ab}^2 \tilde{\mathbf{l}}_2, \\ \Sigma_{ab}^{cd} &= l_{ab} l^{cd} - \tilde{l}_{ab} \tilde{l}^{cd}, \quad \Pi_{ab}^{cd} = I_{ab}^{cd} - \Sigma_{ab}^{cd}, \\ \mathbf{Z}_\mu \cdot \Sigma_{ab} &= 0, \quad \mathbf{Z}_\mu \cdot \Pi_{ab} = Z_\mu^{ab}.\end{aligned}\quad (112)$$

Clearly Σ_{ab} and Π_{ab} become projection operators in the sense that

$$\begin{aligned}\Sigma_{ab} \cdot \Sigma^{cd} &= \frac{1}{2} \Sigma_{ab}^{mn} \Sigma_{mn}^{cd} = \Sigma_{ab}^{cd}, \\ \Pi_{ab} \cdot \Pi^{cd} &= \frac{1}{2} \Pi_{ab}^{mn} \Pi_{mn}^{cd} = \Pi_{ab}^{cd}, \\ \Sigma_{ab} \cdot \Pi^{cd} &= 0.\end{aligned}\quad (113)$$

Now we can express the Einstein-Hilbert action as

$$\begin{aligned}S[e_a^\mu, A_\mu, B_\mu, \mathbf{Z}_\mu] &= \frac{1}{16\pi G_N} \int e \left\{ \mathbf{g}_{\mu\nu} \cdot \mathbf{R}^{\mu\nu} \right. \\ &\quad \left. + \lambda(\mathbf{l}^2 - 1) + \tilde{\lambda}(\tilde{\mathbf{l}} \cdot \tilde{\mathbf{l}}) + \lambda_\mu(\mathbf{l} \cdot \mathbf{Z}^\mu) + \tilde{\lambda}_\mu(\tilde{\mathbf{l}} \cdot \mathbf{Z}^\mu) \right\} d^4x, \\ \mathbf{R}_{\mu\nu} &= \hat{\mathbf{R}}_{\mu\nu} + (\hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu) + \mathbf{Z}_\mu \times \mathbf{Z}_\nu \\ &= (\mathcal{D}_\mu \bar{A}_\nu - \mathcal{D}_\nu \bar{A}_\mu) \mathbf{l} - (\mathcal{D}_\mu B_\nu - \mathcal{D}_\nu B_\mu) \tilde{\mathbf{l}} \\ &\quad + (\hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu),\end{aligned}\quad (114)$$

where λ 's are the Lagrange multipliers. From this we get the following equations of motion

$$\begin{aligned}\delta e_{\mu c}; \quad (e_a^\mu e_b^\nu) [(\mathcal{D}^\nu \bar{A}^\rho - \mathcal{D}^\rho \bar{A}^\nu) l_{ab} \\ - (\mathcal{D}^\nu B^\rho - \mathcal{D}^\rho B^\nu) \tilde{l}_{ab} + (\hat{D}^\nu \mathbf{Z}^\rho - \hat{D}^\rho \mathbf{Z}^\nu) \cdot \Pi_{ab}] e_{\rho c} \\ = 0, \\ \delta A_\nu; \quad \nabla_\mu (e_a^\mu e_b^\nu l^{ab}) + \mathbf{l} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\ \delta B_\nu; \quad \nabla_\mu (e_a^\mu e_b^\nu \tilde{l}^{ab}) + \tilde{\mathbf{l}} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\ \delta \mathbf{Z}_\nu; \quad \hat{\mathcal{D}}_\mu (e_a^\mu e_b^\nu \Pi^{ab}) + (e_a^\mu e_b^\nu) [(\mathbf{Z}_\mu \times \mathbf{l}) l^{ab} \\ - (\mathbf{Z}_\mu \times \tilde{\mathbf{l}}) \tilde{l}^{ab}] = 0. \\ \hat{\mathcal{D}}_\mu = \nabla_\mu + \hat{\Gamma}_\mu \times.\end{aligned}\quad (115)$$

Notice that, using the isometry (46), we can combine the last three equations into a single equation,

$$\mathcal{D}_\mu \mathbf{g}^{\mu\nu} = 0. \quad (116)$$

But this is precisely the second equation of (106), which confirms that (115) is equivalent to (106).

To clarify the meaning of the above equation we define the restricted metric $\hat{\mathbf{g}}_{\mu\nu}$ decomposing $\mathbf{g}_{\mu\nu}$

$$\begin{aligned}\mathbf{g}_{\mu\nu} &= \hat{\mathbf{g}}_{\mu\nu} + \mathbf{G}_{\mu\nu}, \\ \hat{\mathbf{g}}_{\mu\nu} &= e_\mu^a e_\nu^b \Sigma_{ab} = G_{\mu\nu} \mathbf{l} - \tilde{G}_{\mu\nu} \tilde{\mathbf{l}}, \\ \mathbf{G}_{\mu\nu} &= e_\mu^a e_\nu^b \Pi_{ab} = G_{\mu\nu}^1 \mathbf{l}_1 - \tilde{G}_{\mu\nu}^1 \tilde{\mathbf{l}}_1 + G_{\mu\nu}^2 \mathbf{l}_2 - \tilde{G}_{\mu\nu}^2 \tilde{\mathbf{l}}_2, \\ G_{\mu\nu} &= e_\mu^a e_\nu^b l_{ab}, \quad \tilde{G}_{\mu\nu} = e_\mu^a e_\nu^b \tilde{l}_{ab}, \\ G_{\mu\nu}^1 &= e_\mu^a e_\nu^b l_{ab}^1, \quad \tilde{G}_{\mu\nu}^1 = e_\mu^a e_\nu^b \tilde{l}_{ab}^1, \\ G_{\mu\nu}^2 &= e_\mu^a e_\nu^b l_{ab}^2, \quad \tilde{G}_{\mu\nu}^2 = e_\mu^a e_\nu^b \tilde{l}_{ab}^2.\end{aligned}\quad (117)$$

Notice that

$$\begin{aligned}\tilde{G}_{\mu\nu} &= \frac{1}{2} \epsilon_{abcd} e_\mu^a e_\nu^b l^{cd} = \frac{1}{2} \epsilon_{\mu\nu cd} l^{cd} \\ &= \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{\rho\sigma} = G_{\mu\nu}^d, \\ \tilde{G}_{\mu\nu}^1 &= G_{\mu\nu}^{1d}, \quad \tilde{G}_{\mu\nu}^2 = G_{\mu\nu}^{2d}.\end{aligned}\quad (118)$$

Clearly the two two-forms $G_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ can be viewed to represent the restricted metric which are dual to each other. With this (115) has the following compact expression

$$\begin{aligned}G_{\mu\nu}(\mathcal{D}^\nu \bar{A}^\rho - \mathcal{D}^\rho \bar{A}^\nu) - \tilde{G}_{\mu\nu}(\mathcal{D}^\nu B^\rho - \mathcal{D}^\rho B^\nu) \\ + \mathbf{G}_{\mu\nu} \cdot (\hat{D}^\nu \mathbf{Z}^\rho - \hat{D}^\rho \mathbf{Z}^\nu) = 0, \\ \nabla_\mu G^{\mu\nu} + \mathbf{l} \cdot (\mathbf{Z}_\mu \times \mathbf{G}^{\mu\nu}) = 0, \\ \nabla_\mu \tilde{G}^{\mu\nu} + \tilde{\mathbf{l}} \cdot (\mathbf{Z}_\mu \times \mathbf{G}^{\mu\nu}) = 0, \\ \hat{\mathcal{D}}_\mu \mathbf{G}^{\mu\nu} + \mathbf{Z}_\mu \times \hat{\mathbf{g}}^{\mu\nu} = 0,\end{aligned}\quad (119)$$

or equivalently

$$\begin{aligned}G_{\mu\nu}(\mathcal{D}^\nu \bar{A}^\rho - \mathcal{D}^\rho \bar{A}^\nu) - \tilde{G}_{\mu\nu}(\mathcal{D}^\nu B^\rho - \mathcal{D}^\rho B^\nu) \\ + G_{\mu\nu}^i(\mathcal{D}^\nu Z_i^\rho - \mathcal{D}^\rho Z_i^\nu) - \tilde{G}_{\mu\nu}^i(\mathcal{D}^\nu \tilde{Z}_i^\rho - \mathcal{D}^\rho \tilde{Z}_i^\nu) = 0, \\ \nabla_\mu G^{\mu\nu} + \epsilon_{ij}(Z_\mu^i G_j^{\mu\nu} - \tilde{Z}_\mu^i \tilde{G}_j^{\mu\nu}) = 0, \\ \nabla_\mu \tilde{G}^{\mu\nu} + \epsilon_{ij}(Z_\mu^i \tilde{G}_j^{\mu\nu} + \tilde{Z}_\mu^i G_j^{\mu\nu}) = 0, \\ \nabla_\mu G_i^{\mu\nu} - \epsilon_{ij}(\bar{A}_\mu G_j^{\mu\nu} - B_\mu \tilde{G}_j^{\mu\nu} - Z_\mu^j G^{\mu\nu} + \tilde{Z}_\mu^j \tilde{G}^{\mu\nu}) \\ = 0, \\ \nabla_\mu \tilde{G}_i^{\mu\nu} - \epsilon_{ij}(\bar{A}_\mu \tilde{G}_j^{\mu\nu} + B_\mu G_j^{\mu\nu} - Z_\mu^j \tilde{G}^{\mu\nu} - \tilde{Z}_\mu^j G^{\mu\nu}) \\ = 0. \\ (i, j = 1, 2, \quad \epsilon_{12} = -\epsilon_{21} = 1)\end{aligned}\quad (120)$$

This suggests that the valence connection \mathbf{Z}_μ plays the role of the gravitational source of the restricted metric.

In 3-dimensional notation we have

$$\begin{aligned}\hat{\mathbf{g}}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} \\ \hat{e}_{\mu\nu} \end{pmatrix}, \quad \mathbf{G}_{\mu\nu} = \begin{pmatrix} \vec{M}_{\mu\nu} \\ \vec{E}_{\mu\nu} \end{pmatrix}, \\ \mathbf{g}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} + \vec{M}_{\mu\nu} \\ \hat{e}_{\mu\nu} + \vec{E}_{\mu\nu} \end{pmatrix}, \\ \hat{m}_{\mu\nu} &= G_{\mu\nu} \hat{n}, \quad \hat{e}_{\mu\nu} = \tilde{G}_{\mu\nu} \hat{n}, \\ \vec{M}_{\mu\nu} &= G_{\mu\nu}^1 \hat{n}_1 + G_{\mu\nu}^2 \hat{n}_2, \\ \vec{E}_{\mu\nu} &= \tilde{G}_{\mu\nu}^1 \hat{n}_1 + \tilde{G}_{\mu\nu}^2 \hat{n}_2,\end{aligned}\quad (121)$$

so that the Einstein-Hilbert action (114) acquires the following form

$$\begin{aligned}S[e_a^\mu, A_\mu, B_\mu, Z_\mu^i, \tilde{Z}_\mu^i] \\ = \frac{1}{16\pi G_N} \int e \left\{ G_{\mu\nu}(\mathcal{D}^\mu \bar{A}^\nu - \mathcal{D}^\nu \bar{A}^\mu) \right. \\ \left. - \tilde{G}_{\mu\nu}(\mathcal{D}^\mu B^\nu - \mathcal{D}^\nu B^\mu) + G_{\mu\nu}^i(\mathcal{D}^\mu Z_i^\nu - \mathcal{D}^\nu Z_i^\mu) \right. \\ \left. - \tilde{G}_{\mu\nu}^i(\mathcal{D}^\mu \tilde{Z}_i^\nu - \mathcal{D}^\nu \tilde{Z}_i^\mu) \right\} d^4x.\end{aligned}\quad (122)$$

From this we can reproduce (120). This completes the A_2 decomposition (the space-like decomposition) of Einstein's theory.

B. B_2 (Light-like) Decomposition

We can repeat the same procedure with the B_2 isometry (the null isometry) to obtain the desired decomposition of Einstein's equation. With the Einstein-Hilbert action

$$S[e_a^\mu, \Gamma_\mu, \tilde{\Gamma}_\mu, \mathbf{Z}_\mu] = \frac{1}{16\pi G_N} \int e \left\{ \mathbf{g}_{\mu\nu} \cdot \mathbf{R}_{\mu\nu} + \lambda \mathbf{j}^2 + \tilde{\lambda}(\mathbf{j} \cdot \tilde{\mathbf{j}}) + \lambda_\mu(\mathbf{k} \cdot \mathbf{Z}^\mu) + \tilde{\lambda}_\mu(\tilde{\mathbf{k}} \cdot \mathbf{Z}^\mu) \right\} d^4x,$$

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= \hat{\mathbf{R}}_{\mu\nu} + (\hat{D}_\mu \mathbf{Z}_\nu - \hat{D}_\nu \mathbf{Z}_\mu) + \mathbf{Z}_\mu \times \mathbf{Z}_\nu \\ &= (\mathcal{D}_\mu K_\nu - \mathcal{D}_\nu K_\mu) \mathbf{j} - (\mathcal{D}_\mu \tilde{K}_\nu - \mathcal{D}_\nu \tilde{K}_\mu) \tilde{\mathbf{j}} \\ &\quad + (\mathcal{D}_\mu J_\nu - \mathcal{D}_\nu J_\mu) \mathbf{k} - (\mathcal{D}_\mu \tilde{J}_\nu - \mathcal{D}_\nu \tilde{J}_\mu) \tilde{\mathbf{k}} \\ &\quad + (\mathcal{D}_\mu L_\nu - \mathcal{D}_\nu L_\mu) \mathbf{l} - (\mathcal{D}_\mu \tilde{L}_\nu - \mathcal{D}_\nu \tilde{L}_\mu) \tilde{\mathbf{l}}, \end{aligned} \quad (123)$$

we get following equations of motion

$$\begin{aligned} \delta e_{\mu c} ; (e_a^\mu e_b^\nu) &\left[(\mathcal{D}^\nu K^\rho - \mathcal{D}^\rho K^\nu) j_{ab} - (\mathcal{D}^\nu \tilde{K}^\rho - \mathcal{D}^\rho \tilde{K}^\nu) \tilde{j}_{ab} + \mathbf{Z}^{\nu\rho} \cdot \mathbf{\Pi}_{ab} \right] e_{\rho c} = 0, \\ \delta \Gamma_\nu ; \nabla_\mu (e_a^\mu e_b^\nu j^{ab}) &+ \mathbf{j} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\ \delta \tilde{\Gamma}_\nu ; \nabla_\mu (e_a^\mu e_b^\nu \tilde{j}^{ab}) &+ \tilde{\mathbf{j}} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) = 0, \\ \delta \mathbf{Z}_\nu ; \hat{\mathcal{D}}_\mu (e_a^\mu e_b^\nu \mathbf{\Pi}^{ab}) &+ \mathbf{Z}_\mu \times (e_a^\mu e_b^\nu) (k^{ab} \mathbf{j} - \tilde{k}^{ab} \tilde{\mathbf{j}}) \\ &= (e_a^\mu e_b^\nu) (j^{ab} \hat{D}_\mu \mathbf{k} - \tilde{j}^{ab} \hat{D}_\mu \tilde{\mathbf{k}}), \end{aligned} \quad (124)$$

where now

$$\begin{aligned} \Sigma_{ab} &= j_{ab} \mathbf{k} - \tilde{j}_{ab} \tilde{\mathbf{k}}, \\ \mathbf{\Pi}_{ab} &= k_{ab} \mathbf{j} - \tilde{k}_{ab} \tilde{\mathbf{j}} + l_{ab} \mathbf{l} - \tilde{l}_{ab} \tilde{\mathbf{l}} = \mathbf{I}_{ab} - \Sigma_{ab}, \\ \mathbf{Z}_\mu \cdot \Sigma^{ab} &= 0, \quad \mathbf{Z}_\mu \cdot \mathbf{\Pi}^{ab} = Z_\mu^{ab}. \end{aligned} \quad (125)$$

But notice that here $\mathbf{\Pi}_{ab}$ and Σ_{ab} do not make projection operators, because

$$\mathbf{\Pi}_{ab} \cdot \Sigma^{cd} = k_{ab} j^{cd} - \tilde{k}_{ab} \tilde{j}^{cd} \neq 0. \quad (126)$$

Now, again we can combine the last three equations of (124) into a single equation with the isometry (74),

$$\mathcal{D}_\mu \mathbf{g}^{\mu\nu} = 0.$$

This confirms that (124) is equivalent to (106), which tells that (123) describes the Einstein's gravity.

Now, with

$$\begin{aligned} \mathbf{g}_{\mu\nu} &= \hat{\mathbf{g}}_{\mu\nu} + \mathbf{G}_{\mu\nu}, \\ \hat{\mathbf{g}}_{\mu\nu} &= e_\mu^a e_\nu^b \Sigma^{ab} = \mathcal{J}_{\mu\nu} \mathbf{k} - \tilde{\mathcal{J}}_{\mu\nu} \tilde{\mathbf{k}}, \\ \mathbf{G}_{\mu\nu} &= e_\mu^a e_\nu^b \mathbf{\Pi}^{ab} = \mathcal{K}_{\mu\nu} \mathbf{j} - \tilde{\mathcal{K}}_{\mu\nu} \tilde{\mathbf{j}} + \mathcal{L}_{\mu\nu} \mathbf{l} - \tilde{\mathcal{L}}_{\mu\nu} \tilde{\mathbf{l}}, \end{aligned}$$

$$\begin{aligned} \mathcal{J}_{\mu\nu} &= e_\mu^a e_\nu^b j_{ab}, & \tilde{\mathcal{J}}_{\mu\nu} &= e_\mu^a e_\nu^b \tilde{j}_{ab}, \\ \mathcal{K}_{\mu\nu} &= e_\mu^a e_\nu^b k_{ab}, & \tilde{\mathcal{K}}_{\mu\nu} &= e_\mu^a e_\nu^b \tilde{k}_{ab}, \\ \mathcal{L}_{\mu\nu} &= e_\mu^a e_\nu^b l_{ab}, & \tilde{\mathcal{L}}_{\mu\nu} &= e_\mu^a e_\nu^b \tilde{l}_{ab}, \end{aligned} \quad (127)$$

the equation (124) is written as

$$\begin{aligned} \mathcal{J}_{\mu\nu} (\mathcal{D}^\nu K^\rho - \mathcal{D}^\rho K^\nu) - \tilde{\mathcal{J}}_{\mu\nu} (\mathcal{D}^\nu \tilde{K}^\rho - \mathcal{D}^\rho \tilde{K}^\nu) \\ + \mathbf{G}_{\mu\nu} \cdot \mathbf{Z}^{\nu\rho} &= 0, \\ \nabla_\mu \mathcal{J}^{\mu\nu} + \mathbf{j} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) &= 0, \\ \nabla_\mu \tilde{\mathcal{J}}^{\mu\nu} + \tilde{\mathbf{j}} \cdot (\mathbf{Z}_\mu \times \mathbf{g}^{\mu\nu}) &= 0, \\ \hat{\mathcal{D}}_\mu \mathbf{G}^{\mu\nu} + \mathbf{Z}_\mu \times (\mathcal{K}^{\mu\nu} \mathbf{j} - \tilde{\mathcal{K}}^{\mu\nu} \tilde{\mathbf{j}}) \\ &= -\mathcal{J}^{\mu\nu} \hat{D}_\mu \mathbf{k} + \tilde{\mathcal{J}}^{\mu\nu} \hat{D}_\mu \tilde{\mathbf{k}}, \end{aligned} \quad (128)$$

or equivalently

$$\begin{aligned} \mathcal{J}_{\mu\nu} (\mathcal{D}^\nu K^\rho - \mathcal{D}^\rho K^\nu) - \tilde{\mathcal{J}}_{\mu\nu} (\mathcal{D}^\nu \tilde{K}^\rho - \mathcal{D}^\rho \tilde{K}^\nu) \\ + \mathcal{K}_{\mu\nu} (\mathcal{D}^\nu J^\rho - \mathcal{D}^\rho J^\nu) - \tilde{\mathcal{K}}_{\mu\nu} (\mathcal{D}^\nu \tilde{J}^\rho - \mathcal{D}^\rho \tilde{J}^\nu) \\ + \mathcal{L}_{\mu\nu} (\mathcal{D}^\nu L^\rho - \mathcal{D}^\rho L^\nu) - \tilde{\mathcal{L}}_{\mu\nu} (\mathcal{D}^\nu \tilde{L}^\rho - \mathcal{D}^\rho \tilde{L}^\nu) &= 0, \\ \nabla_\mu \mathcal{J}^{\mu\nu} - L_\mu \tilde{\mathcal{J}}^{\mu\nu} - \tilde{L}_\mu \mathcal{J}^{\mu\nu} + J_\mu \tilde{\mathcal{L}}^{\mu\nu} + \tilde{J}_\mu \mathcal{L}^{\mu\nu} &= 0, \\ \nabla_\mu \tilde{\mathcal{J}}^{\mu\nu} + L_\mu \mathcal{J}^{\mu\nu} - \tilde{L}_\mu \tilde{\mathcal{J}}^{\mu\nu} - J_\mu \mathcal{L}^{\mu\nu} + \tilde{J}_\mu \tilde{\mathcal{L}}^{\mu\nu} &= 0, \\ \nabla_\mu \mathcal{K}^{\mu\nu} + L_\mu \tilde{\mathcal{K}}^{\mu\nu} + \tilde{L}_\mu \mathcal{K}^{\mu\nu} &= K_\mu \tilde{\mathcal{L}}^{\mu\nu} + \tilde{K}_\mu \mathcal{L}^{\mu\nu}, \\ \nabla_\mu \tilde{\mathcal{K}}^{\mu\nu} - L_\mu \mathcal{K}^{\mu\nu} + \tilde{L}_\mu \tilde{\mathcal{K}}^{\mu\nu} &= -K_\mu \mathcal{L}^{\mu\nu} + \tilde{K}_\mu \tilde{\mathcal{L}}^{\mu\nu}, \\ \nabla_\mu \mathcal{L}^{\mu\nu} - J_\mu \tilde{\mathcal{K}}^{\mu\nu} - \tilde{J}_\mu \mathcal{K}^{\mu\nu} &= -K_\mu \tilde{\mathcal{J}}^{\mu\nu} - \tilde{K}_\mu \mathcal{J}^{\mu\nu}, \\ \nabla_\mu \tilde{\mathcal{L}}^{\mu\nu} + J_\mu \mathcal{K}^{\mu\nu} - \tilde{J}_\mu \tilde{\mathcal{K}}^{\mu\nu} \\ &= K_\mu \mathcal{J}^{\mu\nu} - \tilde{K}_\mu \tilde{\mathcal{J}}^{\mu\nu}. \end{aligned} \quad (129)$$

Remember that $\mathcal{J}_{\mu\nu}$, $\mathcal{K}_{\mu\nu}$, $\mathcal{L}_{\mu\nu}$ and $\tilde{\mathcal{J}}_{\mu\nu}$, $\tilde{\mathcal{K}}_{\mu\nu}$, $\tilde{\mathcal{L}}_{\mu\nu}$ are dual to each other. Here again the valence connection becomes the gravitational source of the restricted metric.

In 3-dimensional notation we have

$$\begin{aligned} \hat{\mathbf{g}}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} \\ \hat{e}_{\mu\nu} \end{pmatrix}, \quad \mathbf{G}_{\mu\nu} = \begin{pmatrix} \vec{M}_{\mu\nu} \\ \vec{E}_{\mu\nu} \end{pmatrix}, \\ \mathbf{g}_{\mu\nu} &= \begin{pmatrix} \hat{m}_{\mu\nu} + \vec{M}_{\mu\nu} \\ \hat{e}_{\mu\nu} + \vec{E}_{\mu\nu} \end{pmatrix}, \\ \hat{m}_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}} (\mathcal{J}_{\mu\nu} \hat{n}_1 + \tilde{\mathcal{J}}_{\mu\nu} \hat{n}_2) = \hat{n}_3 \times \hat{e}_{\mu\nu}, \\ \hat{e}_{\mu\nu} &= \frac{e^{-\lambda}}{\sqrt{2}} (\tilde{\mathcal{J}}_{\mu\nu} \hat{n}_1 - \mathcal{J}_{\mu\nu} \hat{n}_2) = -\hat{n}_3 \times \hat{m}_{\mu\nu}, \\ \vec{M}_{\mu\nu} &= \frac{e^\lambda}{\sqrt{2}} (\mathcal{K}_{\mu\nu} \hat{n}_1 - \tilde{\mathcal{K}}_{\mu\nu} \hat{n}_2) + \mathcal{L}_{\mu\nu} \hat{n}_3, \\ \vec{E}_{\mu\nu} &= \frac{e^\lambda}{\sqrt{2}} (\tilde{\mathcal{K}}_{\mu\nu} \hat{n}_1 + \mathcal{K}_{\mu\nu} \hat{n}_2) + \tilde{\mathcal{L}}_{\mu\nu} \hat{n}_3, \end{aligned} \quad (130)$$

so that the Einstein-Hilbert action (123) is expressed as

$$\begin{aligned} S[e_a^\mu, K_\mu, \tilde{K}_\mu, J_\mu, \tilde{J}_\mu, L_\mu, \tilde{L}_\mu] \\ = \frac{1}{16\pi G_N} \int e \left\{ \mathcal{J}_{\mu\nu} (\mathcal{D}^\mu K^\nu - \mathcal{D}^\nu K^\mu) \right. \end{aligned}$$

$$\begin{aligned}
& -\tilde{\mathcal{J}}_{\mu\nu}(\mathcal{D}^\mu \tilde{K}^\nu - \mathcal{D}^\nu \tilde{K}^\mu) + \mathcal{K}_{\mu\nu}(\mathcal{D}^\mu J^\nu - \mathcal{D}^\nu J^\mu) \\
& -\tilde{\mathcal{K}}_{\mu\nu}(\mathcal{D}^\mu \tilde{J}^\nu - \mathcal{D}^\nu \tilde{J}^\mu) + \mathcal{L}_{\mu\nu}(\mathcal{D}^\mu L^\nu - \mathcal{D}^\nu L^\mu) \\
& -\tilde{\mathcal{L}}_{\mu\nu}(\mathcal{D}^\mu \tilde{L}^\nu - \mathcal{D}^\nu \tilde{L}^\mu) \Big\} d^4x. \quad (131)
\end{aligned}$$

From this we can reproduce (129). This completes the B_2 decomposition (the light-like decomposition) of Einstein's theory.

V. RESTRICTED GRAVITY

So far our analysis has been mainly on mathematical formalism, and one might wonder what is the physics behind it. The physical motivation behind the mathematical formalism is that we can simplify the Einstein's gravitation and obtain a restricted gravity which can describe the core dynamics of Einstein's theory without compromising the general invariance. In particular the Abelian decomposition allows us to describe the dynamical degrees of Einstein's theory by a spin-one Abelian gauge field.

A common difficulty in quantum gravity and in non-Abelian quantum gauge theory is the highly non-linear self interaction. In gauge theory one can simplify this non-linear interaction by separating the gauge covariant valence part from the Abelian part of the potential and making the Abelian projection to obtain the restricted gauge theory [9, 10]. Here we can simplify Einstein's theory exactly the same way, treating Einstein's theory as a gauge theory of Lorentz group and making Abelian projection, actually two of them, to obtain the restricted gravity. And this restricted gravity presents us a surprising result that the graviton could be described (not only by the spin-two metric field but also) by a photon-like spin-one field.

To understand this we have to understand the restricted gravity first. To do that notice that the above Abelian decomposition is independent of the gauge. More importantly the valence part can be viewed as the Lorentz covariant gravitational source of the Abelian part. So we can remove the valence part without compromising the general invariance, just as we can switch off any gravitational source interacting with gravity to obtain the pure Einstein's theory. This Abelian projection gives us the restricted gravity. And of course we have two restricted gravities, the A_2 gravity and B_2 gravity.

A. A_2 Gravity

Consider the A_2 decomposition first. Clearly (115) tells that the valence connection \mathbf{Z}_μ behaves as the Lorentz covariant gravitational source which couple to the restricted connection, so that we can always put

$\mathbf{Z}_\mu = 0$ without compromising the general invariance (or equivalently the Lorentz gauge invariance). Now, with $\mathbf{Z}_\mu = 0$, (119) is reduced to

$$\begin{aligned}
& G_{\mu\nu}(\partial^\nu \bar{A}^\rho - \partial^\rho \bar{A}^\nu) - \tilde{G}_{\mu\nu}(\partial^\nu B^\rho - \partial^\rho B^\nu) = 0, \\
& \nabla_\mu G^{\mu\nu} = 0, \quad \nabla_\mu \tilde{G}^{\mu\nu} = 0, \\
& \hat{\mathcal{D}}_\mu \mathbf{G}^{\mu\nu} = 0. \quad (132)
\end{aligned}$$

This provides the equations of motion for the A_2 gravity.

To understand the physics behind (132) notice that the first and last equations are the first order differential equations, so that they do not describe the dynamical (i.e., propagating) graviton. They are the constraint equations which determine the connection in terms of the metric. But remarkably the two equations for $G_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ in the middle looks like the free Maxwell's equations. Indeed, since $G_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ are dual to each other, we can express $G_{\mu\nu}$ by one-form potential G_μ

$$G_{\mu\nu} = \nabla_\mu G_\nu - \nabla_\nu G_\mu = \partial_\mu G_\nu - \partial_\nu G_\mu, \quad (133)$$

using the fact $\nabla_\mu \tilde{G}^{\mu\nu} = 0$. Equivalently, we can express $\tilde{G}_{\mu\nu}$ by one-form potential \tilde{G}_μ

$$\tilde{G}_{\mu\nu} = \nabla_\mu \tilde{G}_\nu - \nabla_\nu \tilde{G}_\mu = \partial_\mu \tilde{G}_\nu - \partial_\nu \tilde{G}_\mu, \quad (134)$$

using the fact $\nabla_\mu G^{\mu\nu} = 0$. So we can express the equations of the restricted metric $G_{\mu\nu}$ and $\tilde{G}_{\mu\nu}$ (the 1 and $\tilde{1}$ components of the Lorentz covariant metric $\mathbf{g}_{\mu\nu}$) as a Maxwell-type second order differential equation in terms of the potential G_μ ,

$$\nabla_\mu G^{\mu\nu} = 0, \quad G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu. \quad (135)$$

This is really remarkable and surprising, because this shows that the dynamical part of A_2 gravity can be described by an Abelian gauge theory.

B. B_2 Gravity

Now, exactly the same way we can have the B_2 gravity from the B_2 decomposition. With $\mathbf{Z}_\mu = 0$, we reduce (128) to

$$\begin{aligned}
& \mathcal{J}_{\mu\nu}(\partial^\nu K^\rho - \partial^\rho K^\nu) - \tilde{\mathcal{J}}_{\mu\nu}(\partial^\nu \tilde{K}^\rho - \partial^\rho \tilde{K}^\nu) = 0, \\
& \nabla_\mu \mathcal{J}^{\mu\nu} = 0, \quad \nabla_\mu \tilde{\mathcal{J}}^{\mu\nu} = 0, \\
& \hat{\mathcal{D}}_\mu \mathbf{G}^{\mu\nu} + \mathcal{J}^{\mu\nu} \hat{D}_\mu \mathbf{k} - \tilde{\mathcal{J}}^{\mu\nu} \hat{D}_\mu \tilde{\mathbf{k}} = 0, \quad (136)
\end{aligned}$$

which describes the restricted B_2 gravity.

Here again the first and last equations can be viewed as the constraint equations which determine the connection in terms of the metric. But the two equations for $\mathcal{J}_{\mu\nu}$ and $\tilde{\mathcal{J}}_{\mu\nu}$ in the middle allows us to introduce one-form potential \mathcal{J}_μ for $\mathcal{J}_{\mu\nu}$

$$\mathcal{J}_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu, \quad (137)$$

or $\tilde{\mathcal{J}}_\mu$ for $\tilde{\mathcal{J}}_{\mu\nu}$

$$\tilde{\mathcal{J}}_{\mu\nu} = \partial_\mu \tilde{\mathcal{J}}_\nu - \partial_\nu \tilde{\mathcal{J}}_\mu. \quad (138)$$

With this we can express the equations of the restricted metric $\mathcal{J}_{\mu\nu}$ and $\tilde{\mathcal{J}}_{\mu\nu}$ (the \mathbf{j} and $\tilde{\mathbf{j}}$ components of the Lorentz covariant metric $\mathbf{g}_{\mu\nu}$) as a Maxwell-type second order differential equation in terms of the potential \mathcal{J}_μ ,

$$\nabla_\mu \mathcal{J}^{\mu\nu} = 0, \quad \mathcal{J}_{\mu\nu} = \partial_\mu \mathcal{J}_\nu - \partial_\nu \mathcal{J}_\mu. \quad (139)$$

This shows that the dynamical part of B_2 gravity can also be described by an Abelian gauge theory.

Clearly both (135) and (139) imply that the dynamical field of the restricted gravity is described by a massless spin-one field. But this is nothing but the graviton, because the valence part of the Abelian decomposition simply becomes a gravitational source of the restricted gravity. This means that the graviton can be described by a massless spin-one field. This is a most important outcome of our analysis.

At first thought this view sounds heretical, but actually is not so. First of all, the massless spin-one field has the right degrees of freedom for the graviton. Just as the massless spin-two metric it has two physical degrees. Besides, the metric is not the only field which describes the graviton. Classically the metric is equivalent to tetrads, so that the graviton can also be described by tetrads. And each of the four tetrads becomes a vector. Furthermore, just like the metric, our dynamical fields $G_{\mu\nu}$ and $\mathcal{J}_{\mu\nu}$ are made of tetrads. So it is really not a strange idea to describe the graviton by them. The new (and surprising) thing of our analysis is that they can be expressed by Abelian potentials, through the equation of motion. This leads us to the idea of massless spin-one graviton.

VI. DISCUSSIONS

In this paper we have discussed the Abelian decomposition of Einstein's theory. Imposing proper magnetic isometries to the gravitational connection, we have shown how to decompose the gravitational connection and the curvature tensor into the restricted part of the maximal Abelian subgroup H of Lorentz group G and the valence part of G/H component which plays the role of the Lorentz covariant gravitational source of the restricted connection, without compromising the general invariance.

This tells that the Einstein's theory can be viewed as a theory of the restricted gravity made of the restricted connection in which the valence connection plays the role of the gravitational source of the restricted gravity. We show that there are two different Abelian decompositions of Einstein's theory, light-like A_2 decomposition (the null decomposition) and non light-like B_2 decomposition (the

space/time decomposition), because Lorentz group has two maximal Abelian subgroups.

An important ingredient of the decomposition is the concept of Lorentz covariant four-index metric tensor $\mathbf{g}_{\mu\nu}$ which replaces the role of the two-index space-time metric $g_{\mu\nu}$. We have shown that the metric-compatibility condition of the connection $\nabla_\alpha g_{\mu\nu} = 0$ is replaced by the gauge (and generally) covariant condition $\mathcal{D}_\mu \mathbf{g}^{\mu\nu} = 0$.

From theoretical point of view, the above decomposition of gravitation differs from the Abelian decomposition of non-Abelian gauge theory in one important respect. In gauge theory the fundamental ingredient is the gauge potential, and the decomposition of the potential provides a complete decomposition of the theory. But in gravitation the fundamental field is assumed to be the metric, not the connection (the potential). Because of this the decomposition of the connection gives us the the decomposition of the metric only indirectly, through the equation of motion. It would be very interesting to see if one can actually decompose the metric explicitly, and decompose the Einstein's theory in terms of the metric.

Nevertheless the above decomposition of Einstein's theory has deep implications. First of all, this tells that we can construct a restricted theory of gravitation, actually two of them, which is generally invariant (or equivalently Lorentz gauge invariant) but has fewer physical degrees of freedom than what we have in Einstein's theory. This means that we can separate the Abelian part of gravity which describes the core dynamics of Einstein's theory without compromising the general invariance. More importantly, our analysis shows that we could describe the restricted gravity by an Abelian gauge theory with one-form potential. In other words, our result implies that the graviton can be described by a massless spin-one potential, in stead of the spin-two metric. This has a very important implication, because this point can play a crucial role for us to construct the quantum gravity.

Furthermore, the decomposition makes the topology of Einstein's theory more transparent. Indeed with the Abelian decomposition we can study the topological structures of the theory more easily, because the topological characteristics are imprinted in the magnetic symmetry. For example, the A_2 decomposition makes it clear that the topology of Einstein's theory is closely related to the topology of $SU(2)$ gauge theory. This is natural, because $SU(2)$ forms a subgroup of Lorentz group. This similarity between Einstein's theory and $SU(2)$ gauge theory might be very useful for us to study the gravitomagnetic monopole in Einstein's theory which has the monopole topology $\pi_2(S^2)$ [21, 22].

Perhaps more importantly, this strongly implies that Einstein's theory may have the multiple vacua similar to what we find in $SU(2)$ gauge theory. This turns out to be true. In fact with a proper magnetic isometry we can construct all possible vacuum space-times, and show that

Einstein's theory has exactly the same multiple vacua that we have in $SU(2)$ gauge theory which can be classified by the knot topology $\pi_3(S^3) = \pi_3(S^2)$ [23].

This could have a far reaching consequence. Just as in $SU(2)$ gauge theory, the multiple vacua in Einstein's theory can be unstable against quantum fluctuation. And there is a real possibility that Einstein's theory may admit the gravito-instantons which can connect topologically distinct vacua and thus allow the vacuum tunnelling [23, 24]. Clearly this will have an important implication

in quantum gravity.

The details of the subject with interesting applications will be discussed separately [25].

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